A sufficient condition for super-simplicity of almost-sure theories

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Abstract

In Hill [11], it was shown that an almost-sure theory for a Fraïssé class \( K \) relative to an asymptotic probability measure with “independent sampling” is necessarily super-simple of finite rank with \( D(x=x) = 1 \). In this note, we show that this independent sampling hypothesis is much stronger than necessary. By examining the Aldous-Hoover-Kallenberg (AHK) presentation of an asymptotic probability measure, we find that a certain “ample-fibers” condition is sufficient for super-simplicity. This ample-fibers condition expresses, as a statement about the “hidden” randomness of the AHK-presentation, the assurance that every extension axiom of the generic theory of \( K \) is satisfied.

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Mathematics Subject Classifications: 03C13, 03C15, 03C45, 05A16.

1 Introduction

Since the publication of Fagin [7] and Glebskii et al. [9], it has been known that complete theories, even \( \aleph_0 \)-categorical quantifier-eliminable ones, can arise as the set of sentences of asymptotic probability 1 relative to a collection of probability measures on finite structures; here, these are called almost-sure theories. In the work of Ahlman [2], Koponen [16, 18, 17], and Kruckman [19], it is shown that under various additional hypotheses, \( \aleph_0 \)-categorical super-simple theories with \( D(x=x) = 1 \) (which we will call “simply-minimal theories” here) are very often almost-sure theories – they can be recovered as almost-sure theories by judicious choice of asymptotic probability measures on an underlying Fraïssé class.\(^1\) Conversely, in Hill [11], it is shown that given an “independent-sampling” hypothesis on the asymptotic probability measure, an \( \aleph_0 \)-categorical almost-sure theory must be simply-minimal. Thus, a slightly outrageous conjecture that is implicit in this body of work is the following:

**Conjecture 1.0.1.** Let \( K \) be an algebraically trivial Fraïssé class with generic model (i.e. Fraïssé-limit) \( M \) and generic theory \( T_K = Th(M) \). Then \( T_K \) is an almost-sure theory if and only if it is simply-minimal.

The independent-sampling hypothesis examined in Hill [11] is quite strong, excluding “most” asymptotic probability measures even when we know that independent ones actually exist. In order

\(^1\)Ahlman [2] uses a particularly weak formulation of “asymptotic probability measure” – so that his “almost-sure” is identical to pseudo-finite. The definition employed here and in Hill [11] and Kruckman [19] appears to be significantly stronger.
to consider dealing with the conjecture, then, it appears that one must accommodate these other asymptotic probability measures in one’s thinking. To make this scenario a little more concrete, let us examine the situation among finite graphs and their generic model, the random graph.

Example 1.0.2. For each edge-probability \(0 \leq p \leq 1\), let \(m^p\) be the following family of probability mass functions \(m^p_X : G_X \to [0, 1]\), where for each \(X \subseteq \omega, G_X \) is the set of graphs with vertex set precisely \(X\):

\[ m^p_X(G) = p^e(G) (1 - p)^{\binom{|X|}{2} - e(G)} \]

where \(n = |X|\) and \(e(G)\) is the number of edges in \(G\). Then:

- \(m^p\) is isomorphism-invariant, and if \(X \subseteq Y \subseteq \omega\), then for any \(G \in G_X\), \(m^p_Y(G) = \sum_{G \leq H \in G_Y} m^p_Y(H)\).
- \(T^{m^p} = \{ \varphi : m^p[\varphi] = 1 \}\) is equal to the theory \(T_G\) of the random/Rado graph whenever \(0 < p < 1\), where \(m^p[\varphi] = \lim_{|X| \to \infty} m^p_X[\varphi]\) and \(m^p_X[\varphi] = \sum \{ m^p_X(G) : G \models \varphi \}\).
- Independent sampling: Let \(X, Y \subseteq \omega\) be disjoint, \(z \in \omega \setminus X\), \(G_0 \in K_X\), and \(G \in K_{X \cup \{z\}}\) extending \(G_0\). Then for any \(S \subseteq Y\)

\[
P_{m_{XY}} \left[ \bigwedge_{y \in Y} (H | Xy \models_{G_0} G \iff y \in S) \bigg| G_0 \right] = P_{m_{Xz}} [G | G_0]^{|S|} \cdot (1 - P_{m_{Xz}} [G | G_0])^{|Y| - |S|} 
\]

By the result of Hill [11], these facts about \(m^p\) (when \(0 < p < 1\)) are sufficient to conclude that \(T_G\) is simply-minimal and even that \(G\) contains a measure-1 asymptotic sub-class in the sense of Macpherson-Steinhorn [21] – i.e. definable sets in “typical” finite graphs admit tight size estimates of the flavor of Lang-Weil [20] and Chatzidakis-Macintyre-van den Dries [6].

However, there are many other asymptotic probability measures for \(G\). More precisely, the \(m^p\)’s (including \(p = 0\) and \(p = 1\)) are just the extreme points of a compact, convex space of asymptotic probability measures. For example, if \(0 < p < q < 1\) and \(0 < \alpha < 1\), we might define \(m = \alpha m^p + (1 - \alpha)m^q\) – that is, \(m_X = \alpha m^p_X + (1 - \alpha)m^q_X\) for each \(X \subseteq \omega\). It’s not difficult to verify that \(m\) satisfies the first two conditions above, but it fails on the third count, independent sampling. More generally, if \(\rho\) is a Borel probability measure on \([0, 1]\), then defining each \(m_X\) by

\[ m_X(G) = \int_{[0,1]} m^t_X(G) \, d\rho(t) \]

we recover an asymptotic probability measure \(m\). That is, \(m\) satisfies the first bullet point above. Moreover, if there is an \(0 < \varepsilon < 1/2\) such that \(\rho([\varepsilon, 1 - \varepsilon]) = 1\), then \(m\) also satisfies the second bullet, \(T^m = T_G\), point but still fails on the third.

The natural question here is, “Are these derived, non-extremal asymptotic probability measures still sufficient to induce super-simplicity in the limit model?” The main result of this paper tells us that just the kind of “central concentration” assumption used in defining an asymptotic probability measure from a Borel measure \(\rho\) on \([0, 1]\) is still sufficient to ensure that the limit theory is simply-minimal. In the general case of AHK-presentations, this “central concentration” assumption is rendered as the condition of being amply-fibered. In summary, the main theorem of this paper is the following.
Theorem 3.1.1. Let $K$ be an algebraically trivial Fraïssé class, and let $m$ be an asymptotic probability measure for $K$. Suppose $T^m = T_K$, where $T^m$ is the almost-sure theory of $m$, and $T_K$ is the generic theory of $K$. If $m$ is amply-fibered (in the sense that an Aldous-Hoover-Kallenberg presentation of $m$ is “amply-fibered” in a way that allows us to employ Chernoff bounds under an integral), then $T_K$ is simply-minimal (meaning that $T_K$ is super-simple of finite rank with $D(x=x) = 1$).

1.1 Basic notation and conventions

Throughout this paper, the signature $\text{sig}(\mathcal{L})$ of any language $\mathcal{L}$ under discussion will consist of countably many relation symbols and no constant or function symbols. (It can be shown, as in Hill [10], that under our definition of asymptotic probability measures, almost-sure theories are essentially always algebraically trivial, so there is really no need to consider either constant symbols or function symbols.)

For infinite structures, our notation follows that of Marker [22]. So, for example, we use calligraphic capital letters like $\mathcal{M}$ to denote infinite structures with universe $M$. We denote finite structures using simple capital letters like $A,B,C$, and where appropriate, we identify finite structures with their universes. For an infinite structure $\mathcal{M}$ and a subset $A$ of $M$, $\mathcal{M}\upharpoonright A$ denotes the induced substructure of $\mathcal{M}$ with universe $A$; when no confusion will arise, however, we will often write $A$ instead of $\mathcal{M}\upharpoonright A$. If $B$ is a finite structure and $X \subseteq B$, then similarly, $B\upharpoonright X$ denotes the induced substructure of $B$ with universe $X$. For each $n$, $\text{Def}^n(\mathcal{M})$ is the boolean algebra of $\mathcal{M}$-definable subsets of $M^n$. As usual, a structure $\mathcal{M}$ is called algebraically trivial if $\text{acl}^n(\mathcal{M}) = A$ for all $A \subseteq M$.

We use bold capital letters, especially $K$, to stand for classes of finite structures. Invariably, these are assumed to be closed under isomorphisms and taking induced substructures, and we make the following blanket assumption about any class $K$ under consideration: \textit{For every $n$, the set $\{A \in K : |A| = n\} \cong \text{ is finite.}$} So far, we have discussed notation that is pertinent to almost any discussion of finite structures, but now we turn to some definitions that are specially necessary for our discussion here.

Definition 1.1.1. For each finite set $X \subseteq_{\text{fin}} \omega$, $K_X$ is the set of all structures $A \in K$ with universe exactly $X$. The blanket property above ensures that for each $A \in K_X$, where $X = \{i_0 < \cdots < i_{n-1}\} \subseteq_{\text{fin}} \omega$, there is a quantifier-free formula $\theta_A(x_0, ..., x_{n-1})$ such that for any structure $\mathcal{M}$, if $\{\mathcal{M}\upharpoonright B : B \subseteq_{\text{fin}} M\} \subseteq K$, then $\mathcal{M} \models \theta_A(\bar{a})$ if and only if the map $i_t \mapsto a_t$ is an embedding $A \rightarrow \mathcal{M}$.

If $X \subseteq_{\text{fin}} \omega$, $g \in \text{Sym}(\omega)$, and $A \in K_X$, then $gA$ is the structure with universe $gX$ and interpretations $R^gA = \{g\bar{a} : \bar{a} \in R^A\}$ for each $R \in \text{sig}(\mathcal{L})$; obviously, $g$ is an isomorphism $A \rightarrow gA$.

Definition 1.1.2. Let $K$ be a class of finite $\mathcal{L}$-structures. As usual, $K$ is a Fraïssé class if it has the amalgamation property (AP), the joint-embedding property (JEP), and the heredity property (HP). (These are standard notions – see Hodges [12], for example – so we do not restate them here.) A theorem of Fraïssé [8] asserts that a Fraïssé class $K$ has a countably infinite generic model $\mathcal{M}$ with the following three properties:

- (K-closedness) For every $X \subseteq_{\text{fin}} M$, $\mathcal{M}\upharpoonright X \in K$.
- (K-universality) For every $A \in K$, there is an embedding $A \rightarrow \mathcal{M}$.
- (Ultrahomogeneity) For any $A,B \in K$ and any embedding $f_0 : A \rightarrow \mathcal{M}$, there is an embedding $f : B \rightarrow \mathcal{M}$ such that $f_0 \subseteq f$. 
Moreover, any countable structure with these three properties is isomorphic to \( M \), so \( M \) is usually called the generic model of \( K \) (also called the Fraïssé limit of \( K \)), and the generic theory \( T_K = \text{Th}(M) \) is well-defined in terms of \( K \). Finally, it can be shown that this theory \( T_K \) is \( \aleph_0 \)-categorical and eliminates quantifiers.

**Remark 1.1.3.** For background on simple theories, \( SU \)-rank, and \( D \)-rank, we refer the reader to Wagner [24] and/or Cassanovas [5]. As already suggested, we have invented the following economic terminology: A theory \( T \) is *simply-minimal* if it is super-simple of finite \( SU \)-rank and such that \( D(x=x) = 1 \).

From now on...

We fix an algebraically trivial Fraïssé class \( K \) with generic model \( M \). All discussion in this paper, henceforth, is about these two characters.

## 2 Asymptotic probability measures and AHK-presentations of them

This section is dedicated to the definitions of several important characters: asymptotic probability measures (APMs), \( \text{Sym}(\omega) \)-invariant measures on the space \( S \) of \( L \)-structures with universe \( \omega \), and Aldous-Hoover-Kallenberg-presentations of invariant measures. By showing/noting that invariant measures and APMs are essentially the same thing, we find that the later also have AHK-presentations, so we can use AHK-presentations to study almost-sure theories, which are defined relative to APMs.

### 2.1 Asymptotic probability measures and \( \text{Sym}(\omega) \)-invariant measures

**Definition 2.1.1.** An asymptotic probability measure (APM) for \( K \) is a family \( m = (m_X)_{X \subseteq \text{fin} \omega} \) such that for each \( X \subseteq \text{fin} \omega \):

- \( m_X : K_X \to [0,1] \) is a probability mass function – i.e. \( \sum_{A \in K_X} m_X(A) = 1 \).
- For each \( g \in \text{Sym}(\omega) \), for each \( A \in K_X, m_gX(gA) = m_X(A) \).
- If \( X \subseteq Y \subseteq \text{fin} \omega \), then for each \( A \in K_X, m_X(A) = \sum_{B \in K_Y} m_Y(B) \).

For a sentence \( \varphi \) of \( L \), we set \( m_X[\varphi] = \sum \{ m_X(A) : A \vDash \varphi \} \) for each \( X \subseteq \text{fin} \omega \) and \( m[\varphi] = \lim_{|X| \to \infty} m_X[\varphi] \) (which may fail to converge). We say that \( K \) has the 0,1-law with respect to \( m \) if \( T^m = \{ \varphi : m[\varphi] = 1 \} \) is a complete theory. We say that \( K \) is a 0,1-class via \( m \) just in case \( T^m = T_K \).

Independent APMs were the key player in Hill [11], and since the present article generalizes that result, we recall the that definition (but it will play little role in this discussion).

**Definition 2.1.2.** Let \( m \) be an APM for \( K \). We say that \( m \) is independent if the following holds:

Let \( X, Y \subseteq \text{fin} \omega \) be disjoint, \( z \in \omega \setminus X, A \in K_X, \) and \( B^* \in K_{XU\{z\}} \) extending \( A \); then

\[
\mathbb{P}_{m_XY} \left[ \bigwedge_{y \in Y} (B | X y \equiv_A B^* \iff y \in S) \bigg| A \right] = \mathbb{P}_{m_Xz} \left[ B^* | A \right]^{S} \cdot (1 - \mathbb{P}_{m_Xz} \left[ B^* | A \right])^{Y - |S|}
\]

for every \( S \subseteq Y \) (\( B \) ranging over \( K_{XY} \)).
Definition 2.1.3. Let $S = S_K$ denote the set of all $\mathcal{L}$-structures $\mathcal{N}$ with universe exactly $\omega$ and such that $\mathcal{N}[X] \in K$ for every $X \subseteq_{\text{fin}} \omega$. For $X \subseteq_{\text{fin}} \omega$ and $A \in K_X$, let $[A] = \{\mathcal{N} \in S : \mathcal{N}[X] = A\}$; let $B$ be the boolean algebra generated by $\{[A] : A \in K_X, X \subseteq_{\text{fin}} \omega\}$. Then $B$ forms a base of clopen sets for a Cantor topology on $K$. Let $\text{Bor}(S)$ denote the $\sigma$-algebra of Borel sets of $S$ relative to this topology. We observe that $\text{Sym}(\omega)$ acts on $S$ by homeomorphisms.

An invariant measure is a $\sigma$-additive probability measure $\mu : \text{Bor}(S) \to [0,1]$ such that for every $W \in \text{Bor}(S)$ and every $g \in \text{Sym}(\omega)$, $\mu(gW) = \mu(W)$.

Observation 2.1.4. APMs and invariant measures are essentially the same thing:

- Suppose $m$ is an asymptotic probability measure for $K$. We define $\nu_0^m : B \to [0,1]$ by setting $\nu_0^m([A]) = m_X(A)$ whenever $A \in K_X, X \subseteq_{\text{fin}} \omega$. We note that each set $W \in B$ is of the form $[A_0] \cup \cdots \cup [A_m]$, and this, together with the third (marginalization) condition in the definition of an asymptotic probability measure, ensures that $\nu_0^m$ is a finitely-additive probability measure on $B$. Applying the Hahn-Kolmogorov Theorem, $\nu_0^m$ extends to a $\sigma$-additive probability measure $\nu^m : \text{Bor}(S) \to [0,1]$. Finally, the second condition in the definition of an APM, isomorphism-invariance, ensures that $\nu^m$ is $\text{Sym}(\omega)$-invariant.

- Suppose $\mu : \text{Bor}(S) \to [0,1]$ is an invariant measure. We define $b_\mu = (b_\mu^X)_{X \subseteq_{\text{fin}} \omega}$ as follows: For $X \subseteq_{\text{fin}} \omega$ and $A \in K_X$, $b_\mu^X(A) = \mu([A])$. It is straightforward to check that $b_\mu$ is indeed an asymptotic probability measure.

- It is also fairly easy to verify that $b_\mu^m = m$ whenever $m$ is an asymptotic probability measure for $K$ and that $\nu^{b_\mu} = \mu$ whenever $\mu$ is an invariant measure.

2.2 AHK-presentations

The Aldous-Hoover-Kallenberg theorem (proved in various forms independently in Aldous [3], Hoover [13], and Kallenberg [14]) allows one to re-express an invariant measure on $S$ as a random process using independent random sources indexed by finite subsets of $\omega$. In particular, this allows one to present an APM as a family of integrals (against product measures on powers of $\omega$) as follows:

- Let $\xi = (\xi_X)_{X \subseteq_{\text{fin}} \omega}$ be an iid ensemble of $[0,1]$-valued random variables, where for each $X \subseteq_{\text{fin}} \omega$, $\xi_X$ is uniformly distributed. For $\mathcal{F} \subseteq \mathcal{P}_{\text{fin}}(\omega)$, $\xi_{\mathcal{F}}$ is the restricted ensemble $(\xi_X)_{X \in \mathcal{F}}$. Writing $\lambda$ for the Lebesgue measure on $[0,1]$, we have $\mathbb{P}(\xi_X \in [a,b]) = \lambda([a,b])$ whenever $0 \leq a < b \leq 1$ and $X \subseteq_{\text{fin}} \omega$.

Suppose $X,Y \subseteq_{\text{fin}} \omega$ are disjoint. Then $\mathcal{P}_X(Y) = \{S \subseteq X \cup Y : S \nsubseteq X\}$. For $X \subseteq_{\text{fin}} \omega$ and $i \in \omega \setminus \mathcal{Q}_i(X) = \{i\} \cup S : S \subseteq X\} = \mathcal{P}_X(\{i\})$. For each $Y \subseteq_{\text{fin}} \omega$, $\lambda_Y = \lambda^{\otimes \mathcal{P}_X(Y)}$ is the product measure. Moreover, if $X,Y \subseteq_{\text{fin}} \omega$ are disjoint, then $\lambda_{Y/X} = \lambda^{\otimes \mathcal{P}_X(Y)}$. We observe that if $Z = X \cup Y$ and $x \in [0,1]^\mathcal{P}(X)$, then the conditional measure $\lambda_Z(-|x)$ is actually the product measure $\lambda^{\otimes \mathcal{P}_X(Y)}$.

We write $[0,1]^*$ as shorthand for the infinite product space $[0,1]^{\mathcal{P}_{\text{fin}}(\omega)}$ and $\lambda^*$ for the product measure on this space.

Theorem 2.2.2 (As in Ackerman-Freer-Kruckman-Patel [1] and Kallenberg [15]). Suppose $\mu : \text{Bor}(S) \to [0,1]$ is an invariant measure. Then $\mu$ admits an AHK-presentation – a family of measurable functions $F_Y = F_Y^\mu : [0,1]^\mathcal{P}(Y) \to K_Y$ ($Y \subseteq_{\text{fin}} \omega$) as follows:
AHK1. For all $X \subseteq Y \subseteq_{\text{fin}} \omega$ and every $g \in \text{Sym}(Y)$, $F_{X}((x_{g^{-1}S})_{S \in \mathcal{P}((gX)}) = gF_{X}(x|_{\mathcal{P}(X)})$ for \( \lambda_{Y} \)-almost every \( x \in [0,1]^{\mathcal{P}(Y)} \).

AHK2. For any $X \subseteq Y \subseteq_{\text{fin}} \omega$, $F_{Y}(x|X) = F_{X}(x|_{\mathcal{P}(X)})$ for \( \lambda_{Y} \)-almost every \( x \in [0,1]^{\mathcal{P}(Y)} \).

AHK3. For every $X \subseteq Y \subseteq_{\text{fin}} \omega$ and every $A \in \mathcal{K}_{X}$, $\mu([A]) = \lambda_{X}(F_{X}^{-1}A)$

A family $F = (F_{Y})_{Y \subseteq_{\text{fin}} \omega}$ satisfying AHK1 and AHK2 will also be called an AHK-system for \( \mathcal{L} \)-structures. (Obviously, the system can exist without a pre-specified invariant measure.) If \( m \) is an asymptotic probability measure, then any AHK-presentation of $\mu^{m}$ is an AHK-presentation of \( m \).

We note that any AHK-system for \( \mathcal{L} \)-structures induces an invariant measure: For if $F_{\bullet}$ is an AHK-system for \( \mathcal{L} \)-structures, then we may define $\nu^{F_{\bullet}}$ as follows: For each $X \subseteq_{\text{fin}} \omega$ and every $A \in \mathcal{K}_{X}$, we set

$$\nu^{F_{\bullet}}_{0}([A]) = \lambda_{X}(F_{X}^{-1}A)$$

As in Observation 2.1.4, $\nu^{F_{\bullet}} : \mathcal{B} \rightarrow [0,1]$ is a $\text{Sym}(\omega)$-invariant pre-measure, and it induces a $\text{Sym}(\omega)$-invariant $\sigma$-additive measure $\nu^{F_{\bullet}} : \text{Bor}(\mathcal{S}) \rightarrow [0,1]$. Finally, one can show that $\nu^{F_{\bullet}} \circ = \mu$ whenever $\mu$ is an invariant measure.

One benefit of using AHK-presentations is that, because they are defined against product measures, conditional probabilities are particularly easy to express – as in the following definition.

**Definition 2.2.3.** Let $F_{\bullet}$ be an AHK-system for \( \mathcal{L} \)-structures. Suppose $Z = X \cup Y$, $x \in [0,1]^{\mathcal{P}(X)}$, and $B \in \mathcal{K}_{Z}$. Then

$$\mathbb{P}(F_{Z}(x, \xi|_{\mathcal{P}_{X}(Y)}) = B \mid x) = \lambda_{Y/X}(\{y \in [0,1]^{\mathcal{P}_{X}(Y)} : F_{Z}(x, y) = B\}).$$

With AHK-presentations, conditional probabilities are easy to write down, and since the random variables in $\xi$ are iid, we also observe that many derived random variables are conditionally independent.

**Fact 2.2.4.** Let $F_{\bullet}$ be an AHK-system for \( \mathcal{L} \)-structures. Let $X \subseteq_{\text{fin}} \omega$ and $C \in \mathcal{K}_{X}$, and let $y_{0} < \cdots < y_{n-1}$ be distinct elements of $\omega \setminus X$. For each $j < n$, let $Y_{j} = X \cup \{y_{j}\}$ and $B_{j} \in \mathcal{K}_{Y_{j}}$ such that $B_{j}|X = C$. Then

$$\mathbb{P}(\bigwedge_{j<n} F_{Y_{j}}(x, \xi|_{\mathcal{Q_{n}}(X)}) = B_{j} \mid x) = \prod_{j<n} \mathbb{P}(F_{Y_{j}}(x, \xi|_{\mathcal{Q_{n}}(X)}) = B_{j} \mid x)$$

for $\lambda_{X}$-almost every $x \in [0,1]^{\mathcal{P}(X)}$.

To conclude this section, we define the notion of an amply-fibered asymptotic probability measure. These are not independent APMs themselves but, in some sense, mixtures of independent APMs that strongly avoid concentrating on any independent APM that fails to capture $T_{K}$.

**Definition 2.2.5.** Let $F_{\bullet}$ be an AHK-system for \( \mathcal{L} \)-structures. We say that $F_{\bullet}$ is amply-fibered (has ample fibres) if for every $Y \subseteq_{\text{fin}} \omega$ and $i \in \omega \setminus Y$ (and $Z = Y \cup \{i\}$), there is an $\varepsilon > 0$ such that for all $B \in \mathcal{K}_{Z}$, if $C = B|Y$, then

$$\lambda_{Y}(\{x \in F_{Y}^{-1}C : \mathbb{P}(F_{Z}(x, \xi|_{\mathcal{Q}(Y)}) = B \mid x) < \varepsilon\}) = 0.$$
3 Ample fibres yield super-simplicity

3.1 Statement and outline of the proof

The necessary definitions are now in place, and in this section, we will prove the main result of this paper:

**Theorem 3.1.1.** Suppose $\mathcal{K}$ is a 0,1-class by way of an asymptotic probability measure $\mathfrak{m}$. If $\mathfrak{m}$ is amply-fibered, then $T_\mathcal{K}$ is simply-minimal.

The proof of Theorem 3.1.1 accounts for the remainder of this section – Subsections 3.2 and 3.3. An outline of the proof is as follows:

- If $\mathfrak{m}$ is amply-fibered, then we can use a version of the Chernoff bound (see Blass-Gurevich [4] or Motwani-Raghavan [23], for example) under an integral sign to show that, with probability tending rapidly to 1, every “1-point extension problem” of structures in $\mathcal{K}$ not only has a solution but a strictly-positive fraction of elements are solutions. This is the project of Subsection 3.2.

- In Subsection 3.3, we assume that $T^\mathfrak{m} = T_\mathcal{K}$ and that $\mathfrak{m}$ is amply-fibered. Using the work of Subsection 3.2, we show how, given an infinite definable set $X \subseteq M$, to construct a probability measure on $\nu : \text{Def}^1(M) \to [0,1]$ and a lower-bound $\beta > 0$ such that $\nu(X') \geq \beta$ whenever $X'$ is a conjugate of $X$. We then use this construction to show that there is essentially no forking models of in $T_\mathcal{K}$.

3.2 Chernoff bounds under the integral

The fact that animates our demonstration is the following fact, which is very well-known in discrete applications of probability theory. After stating it in its usual form, we give another form that will be more convenient in our work here.

**Lemma 3.2.1 (Chernoff bound).** There is a function $\zeta : (0,1) \times (0,1) \to (0,1)$ such that for any $0 < p < 1$ and any $0 < \alpha < 1$, if $X_0, \ldots, X_{n-1}$ are iid Bernoulli($p$)-random variables, then

$$\mathbb{P}[\sum_{i<n} X_i \leq \alpha pn] \leq \zeta(p,\alpha)^n.$$

**Lemma 3.2.1. (Restatement)** For each $n$, define $f_n : (0,1) \times (0,1) \to \mathbb{R}$ by

$$f_n(p,\alpha) = \sum_{k=0}^{\lfloor \alpha pn \rfloor} \binom{n}{k} p^k (1-p)^{n-k}.$$

There is a function $\zeta : (0,1) \times (0,1) \to (0,1)$ such that for any $0 < p < 1$, any $0 < \alpha < 1$, and any $n$, $f_n(p,\alpha) \leq \zeta(p,\alpha)^n$.

The main tool that will be used in Subsection 3.3 is Proposition 3.2.3. We derive the latter as a relatively easy consequence of the following, Lemma 3.2.2.
Lemma 3.2.2. Suppose \( m \) is amply-fibered. Let \( X \subset \fin \omega, B \in K_X, i^* \in \omega \setminus X, \) and \( B' \in K_{X \cup \{i^*\}} \) such that \( B \leq B' \). Let \( \varphi(x, \bar{b}) \) be a quantifier-free formula that isolates the quantifier-free-complete type of \( i^* \) over \( B \) in \( B' \). If \( m_X(B) > 0 \), then there are \( 0 < \beta_0, \zeta < 1 \) such that

\[
\frac{\mathbb{P}_{m_X} \{ A : |\varphi(A, \bar{b})| \leq \beta_0 |Y| \}}{m_X(B)} \leq \zeta^{|Y|}
\]

whenever \( Y \subset \fin \omega \setminus X \).

Proof. By hypothesis, let \( \varepsilon > 0 \) be such that

\[
\lambda_X \left( \left\{ x \in F_X^{-1}B : \mathbb{P}(F_Xi^*(x, \xi_{\rho(x)}) = B' | x) \geq \varepsilon \right\} \right) = 1.
\]

We claim that \( \zeta = \zeta(\varepsilon, 1/2) \) and \( \beta_0 = \varepsilon/2 \) suffice. When \( n = |Y| \), we observe:

\[
\frac{\mathbb{P}_{m_X} \{ A : |\varphi(A, \bar{b})| \leq \beta_0 n \}}{m_X(B)} =
\]

\[
= \lambda_X(F_X^{-1}B)^{-1} \cdot \sum_{k=0}^{\lfloor \beta_0 n \rfloor} \sum_{S \in (Y)} \int_{F_X^{-1}B} \prod_{y \in Y} \mathbb{P} \left( (F_{Xy}(x, \xi_{|\rho(x)}(X)) \cong_{B'} B')^{y \in S} \bigg| x \right) d\lambda_X(x)
\]

\[
\leq \lambda_X(F_X^{-1}B)^{-1} \cdot \int_{F_X^{-1}B} f_n(\varepsilon, 1/2) d\lambda_X(x)
\]

\[
= f_n(\varepsilon, 1/2) \cdot \lambda_X(F_X^{-1}B)^{-1} \cdot \int_{F_X^{-1}B} d\lambda_X(x)
\]

\[
\leq \zeta(\varepsilon, 1/2)^n
\]

as desired. \( \square \)

Proposition 3.2.3. Suppose \( m \) is amply-fibered. Let \( X \subset \fin \omega, B \in K_X, i^* \in \omega \setminus X, \) and \( B' \in K_{X \cup \{i^*\}} \) such that \( B \leq B' \). Let \( \varphi(x, \bar{b}) \) be a quantifier-free formula that isolates the quantifier-free-complete type of \( i^* \) over \( B \) in \( B' \). If \( m_X(B) > 0 \), then there are real numbers \( \beta > 0 \) and \( 0 < \zeta < 1 \) and an integer \( 0 < k < \omega \), such that

\[
\mathbb{P}_{m_Y} \left\{ A \in K_Y : (\exists u \in \text{Emb}(B, A)) |\varphi(A, u\bar{b})| \leq \beta |X| \right\} \in O\left( |X|^k \cdot \zeta^{|X|} \right)
\]

whenever \( Y \subset \fin \omega \) is sufficiently large. (Here, \( \text{Emb}(B, A) \) denotes the set of embeddings \( B \to A \).)

Proof. Let \( 0 < \beta_0, \zeta < 1 \) be as in Lemma 3.2.2 and set \( k = |X| \). Let \( 0 < \beta < 1 \) be such that \( \beta |Y| \leq \beta_0 |Y| - k \) whenever \( |Y| \) is sufficiently large. Then since probability measures are finitely sub-additive, we see that for some \( \alpha > 0 \) and for all large enough \( Y \),

\[
\mathbb{P}_{m_Y} \left\{ A : (\exists u \in \text{Emb}(B, A)) |\varphi(A, u\bar{b})| \leq \beta |Y| \right\} \leq |Y|^k \cdot \frac{\mathbb{P}_{m_Y} \{ A : |\varphi(A, \bar{b})| \leq \beta |Y| \}}{m_X(B)} = \zeta^{-k} \cdot |Y|^k \cdot \zeta^{|Y|}
\]

as required. \( \square \)
3.3 Proof of Theorem 3.1.1

In this subsection, we will use Proposition 3.2.3 to prove that if there is an amply-fibered APM \( m \) for \( K \) such that \( T^m = T_K \), then \( T_K \) is simply-minimal – i.e. there is essentially no forking in models of \( T_K \). To do this, we first formulate two facts – Lemmas 3.3.1 and 3.3.2 – that allow us to show that a given formula does not divide. The author is not aware of a reference for Lemma 3.3.1, but it seems most likely that it was previously known; in any case, it is at least morally present in Macpherson-Steinhorn [21]. The proof of Lemma 3.3.2 is routine, so we omit it.

Lemma 3.3.1. Let \( N \) be an \( \aleph_0 \)-saturated infinite structure, and let \( \varphi(x, \bar{b}, \bar{c}) \in \mathcal{L}(N) \). Suppose there are \( \beta > 0 \) and a finitely-additive probability measure \( \nu : \text{Def}^1(N) \rightarrow [0,1] \) such that

\[
\nu(\varphi(M, \bar{b}, \bar{c})) \geq \beta \text{ whenever } \bar{b} \equiv_{\bar{c}} \bar{b}.
\]

Then \( \varphi(x, \bar{b}, \bar{c}) \) does not 2-divide over \( \bar{c} \).

Proof. For a contradiction, suppose \( \varphi(x, \bar{b}, \bar{c}) \) 2-divides over \( \bar{c} \). Since \( N \) is \( \aleph_0 \)-saturated, there is an \( \bar{c} \)-indiscernible sequence \( (\bar{b}_i)_{i<\omega} \) such that \( \bar{b}_0 = \bar{b} \) and \( \{\varphi(x, \bar{b}_i, \bar{c})\}_{i<\omega} \) is 2-inconsistent – i.e. \( \varphi(N, \bar{b}_i, \bar{c}) \cap \varphi(M, \bar{b}_j, \bar{c}) = \emptyset \) whenever \( i < j < \omega \). Now, let \( N < \omega \) such that \( N\beta > 1 \). Then

\[
1 \geq \nu \left( \bigcup_{i<N} \varphi(M, \bar{b}_i, \bar{c}) \right) = \sum_{i<N} \nu(\varphi(N, \bar{b}_i, \bar{c}) \geq N\beta > 1
\]

which is nonsense. Thus, \( \varphi(x, \bar{b}, \bar{c}) \) does not 2-divide over \( \bar{c} \). \( \square \)

Lemma 3.3.2. Let \( C_0, ..., C_k, ... \) be a chain of finite subsets of \( M \) such that \( \bigcup_k C_k = M \), and suppose that for every \( \varphi \in T_K \), \( \{k : C_k \models \varphi\} \) is co-finite. Let \( \mathcal{U} \) be some non-principal ultrafilter on \( \omega \), and let \( M^* = \prod_k C_k / \mathcal{U} \) with universe \( M^* \).

- Define \( f : M \rightarrow M^* \) as follows: For each \( a \in M \), let \( w^a = (w^a_k)_{k \in \omega} \) be such that \( w^a_k = a \) whenever \( a \in C_k \); set \( f(a) := w^a / \mathcal{U} \). Then, this \( f \) is an elementary embedding.

- We define \( \nu = \nu[C_\bullet, \mathcal{U}] : \text{Def}^1(M) \rightarrow [0,1] \) as follows:

\[
\nu(\varphi(M, \bar{b})) = \lim_{k \rightarrow \mathcal{U}} \frac{|\varphi(C_k, f\bar{b})|}{|C_k|}.
\]

(This is the counting measure induced by \( C_\bullet \) and \( \mathcal{U} \).) Then \( \nu \) is a finitely-additive probability measure on \( \text{Def}^1(M) \).

Our next lemma will allow us to apply Lemmas 3.3.1 and 3.3.2 in the context of an amply-fibered APM. The construction in the proof is somewhat similar to one given in Hill [11], but this one is somewhat easier as the goal here (dealing with just one formula \( \varphi \) and only needing a lower bound) is more modest.

Lemma 3.3.3. Suppose \( m \) is amply-fibered and \( T^m = T_K \). Let \( \varphi(x, \bar{b}, \bar{c}) \in \mathcal{L}(M) \) be a quantifier-free formula that isolates a complete 1-type \( tp(a/\bar{b}c) \), where \( a \notin \bar{b}c \). Then we can choose \( \beta > 0 \) and \( C_0, ..., C_k, ... \) as in the hypotheses of Lemma 3.3.2 so that if \( \nu \) is the counting measure induced by \( C_\bullet \) and \( \mathcal{U} \), then \( \nu \left( \varphi(M, \bar{b}, \bar{c}) \right) \geq \beta \) whenever \( \bar{b} \equiv_{\bar{c}} \bar{b} \).
Proof. Let $B$ be the substructure of $\mathcal{M}$ induced by $\vec{b}$. Let $0 < \beta, \zeta < 1$ be as in Proposition 3.2.3, and let $\varphi_0, \ldots, \varphi_n, \ldots$ be an enumeration of $T_K$. We choose $C_0, \ldots, C_k, \ldots$ as follows:

- Choose $C_0 \subset \text{fin} M$ containing $\vec{r}$ arbitrarily.
- Given $C_0 \subset \cdots \subset C_k$, proceed as follows:
  - Let $\psi = \exists \pi \theta_{C_k}(\vec{x}) \wedge \bigwedge_{i \leq k} \varphi_i$
  - Let $N$ be large enough to ensure that if $|Y| \geq N$, then $\mathbb{P}_{\text{mv}} \{ C : C \models \psi \} \geq 2/3$ and
    \[
    \mathbb{P}_{\text{mv}} \left\{ C : (\forall u \in \text{Emb}(B,C)) \left| \varphi(C,u\vec{b},\vec{r}) \right| \geq \beta |Y| \right\} \geq 2/3
    \]
    - Choosing any $Y \subset \omega$ such that $|Y| \geq N$, we may then choose $C \in \mathcal{K}_Y$ such that $C \models \psi$ and $|\varphi(C,\vec{b}',\vec{r})| \geq \beta |Y|$ whenever $\text{qftp}^{C}(\vec{b}',\vec{r}) = \text{qftp}^{\mathcal{M}}(\vec{b})$.
    - Since $\mathcal{M}$ is $\mathcal{K}$-universal, there is an embedding $w : C \rightarrow \mathcal{M}$, and since $C \models \exists \pi \theta_{C_j}(\vec{x})$, there is an embedding $v : C_k \rightarrow C$. Then $w \circ v : C_k \rightarrow \mathcal{M}$ is an embedding, so as $\mathcal{M}$ is ultrahomogeneous, there is an automorphism $g \in \text{Aut}(\mathcal{M})$ extending $w \circ v$. Finally, we take $C_{k+1} = g^{-1}wC$.

The chain $C_0 \subset \cdots \subset C_k \subset \cdots$ evidently satisfies the requirements of the lemma.

Proof of Theorem 3.1.1. For any theory $T$, if $D(x=x) = k$ for some $k < \omega$, then for every $n$ and every definable set $X \subset N^n$ of a model $\mathcal{N} \models T$, we have $D(X) \leq nk$. Thus, if $D(x=x) = 1$, then $T_K$ is super-simple of finite rank. Thus, it is enough to prove the following:

Claim. Suppose $\mathfrak{m}$ is amply-fibered and $T^\mathfrak{m} = T_K$. Then $D(x=x) = 1$.

Proof of claim. Let $\varphi(x,\vec{b},\vec{r})$ be a quantifier-free formula such that $\varphi(x,\vec{b},\vec{r}) \models \lnot x \in \vec{b}$. Towards a contradiction, suppose $\varphi(x,\vec{b},\vec{r})$ $k$-divides over $\vec{r}$ for some $k \geq 2$. By $\aleph_0$-categoricity and quantifier-elimination, we may assume that $\varphi(x,\vec{b},\vec{r})$ isolates a complete type. Let $(\vec{b}_i)_{i < \omega}$ be a $\vec{r}$-indiscernible sequence such that $\vec{b}_0 = \vec{b}$ and $\{ \varphi(x,\vec{b}_i) \}_{i < \omega}$ is $k$-inconsistent. We may assume that $\{ \varphi(x,\vec{b}_i,\vec{r}) \}_{i < \omega}$ is $\ell$-consistent whenever $\ell < k$. In particular, let $\ell = k-1$, and for each $j < \omega$, let $\vec{r}_j = \vec{b}_j \vec{b}_{j+1} \vec{b}_{j+2} \cdots \vec{b}_{(j+1)\ell-1}$. Also, let $\psi(x,\vec{r}_j,\vec{r}) = \bigwedge_{i < \ell} \varphi(x,\vec{b}_i,\vec{r})$, which is consistent by our minimality assumption on $k$. Then $(\vec{r}_j)_{j < \omega}$ is is a $\vec{r}$-indiscernible sequence, and $\{ \varphi(x,\vec{r}_j,\vec{r}) \}_{j < \omega}$ is $2$-inconsistent. Hence $\psi(x,\vec{r}_0,\vec{r})$ $2$-divides over $\vec{r}$, and it follows that for some $a \in \psi(\mathcal{M},\vec{r}_0,\vec{r})$, $\theta(x,\vec{r}_0,\vec{r}) = \text{qftp}(a/\vec{r}_0,\vec{r})$ also $2$-divides over $\vec{r}$.

One the other hand, by Lemma 3.3.3, we recover $\beta > 0$ and a finitely-additive probability measure $\nu : \text{Def}^1(\mathcal{M}) \rightarrow [0,1]$ such that $\nu(\theta(\mathcal{M},\vec{r},\vec{r})) \geq \beta$ whenever $\vec{r} \equiv_{\mathcal{M}} \vec{r}_0$. By Lemma 3.3.1, it follows that $\theta(x,\vec{r}_0,\vec{r})$ does not $2$-divide over $\vec{r}$ -- a contradiction.

This completes the proof of the theorem.

4 Last thoughts

Our main theorem gives a sufficient condition for simple-minimality of an almost-sure theory that is apparently a weaker assumption than that of Hill [11]. Since the two hypotheses yield similar consequences, it is natural to ask if our weaker hypothesis is actually sufficient for the stronger one. Thus, we make the following conjecture.
Conjecture 4.0.1. If there is an amply-fibered APM m for K such that $T^m = T_K$, then there is an independent APM $m'$ for K such that $T^{m'} = T_K$.

All of the APMs we’ve discussed here and in Hill [11] are elements a Banach space $X$, and it is at least plausible that simple-minimality of $T_K$ can be re-expressed in terms of properties of subsets of $X$. Exactly which properties of $X$ are germane is not entirely clear, but in the following, we attempt to identify them. (It is inspired, in some sense, by the result of Ackerman-Freer-Kruckman-Patel [1] that the ergodic measures are the extreme points of the set of all invariant measures.)

Conjecture 4.0.2. Let $X$ be the Banach space under the total-variation norm generated by bounded finitely-additive $\text{Sym}(\omega)$-invariant functions $f : \mathcal{B} \to \mathbb{R}$ (where $\mathcal{B}$ is the boolean algebra of clopen sets of $S = S_K$ given in Definition 2.1.3). Let $M^0_K$ be the set of APMs m such that $T^m = T_K$, and let $M_K$ be the closure of $M^0_K$. Then $M_K$ is a compact convex subset of $X$, so by the Krein-Milman Theorem, $M_K$ is equal to the closure of the convex hull of its extreme points. We conjecture that:

1. The extreme points of $M_K$ are the independent APMs for $K$ (allowing $m_X(A) = 0$ for some $X \subset_{\text{fin}} \omega$, $A \in K_X$).

2. Let $E^*$ be the set of extreme points m of $M_K$ such that $m_X(A) > 0$ for all $X \subset_{\text{fin}} \omega$ and $A \in K_X$. Then:

(a) $E^*$ is the set of independent APMs m for K such that $T^m = T_K$.

(b) $T_K$ is simply-minimal if and only if $E^*$ is non-empty.

References


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