ON CONSTRAINED GENERIC EXPANSIONS
AND STRUCTURAL RAMSEY THEORY

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*** DRAFT ***

INTRODUCTION

It is reasonably well-known in model theory that expansions of countable countably-categorical structures are closely associated with certain compactly metrizable spaces and, further, that a generic orbit in such a space – in the sense of Baire category – corresponds to an expansion with a particularly well-behaved model theory (relative to the base structure). Results of this kind can be found in [2], [6], [3] and several other publications. The first contribution of this article lies in demonstrating that for any expansion $B$ (by finitely many new relations) of a countable countably-categorical structure $A$, there is canonical $\aleph_0$-categorical “maximally” generic expansion of $A$ constrained by $B$ – an expansion that is Baire-generic in the appropriate space and realizes no configurations that are not already realized in $B$.

The motivating example, for us, of an expansion $B$ of $A$ is the expansion induced by a coloring of copies of a given finite substructure of $A$ – this arises in the formulation and analysis of the Ramsey Property of a class $K$ of finite structures from which $A$ is generated (as, of course, the generic model of $K$). A first application of our analysis of constrained generic expansions is proving an equivalence between the Ramsey Property for $K$ and a natural Generic Infinitary Ramsey Property for the generic model $A$. The latter is formulated in terms of generic colorings and elementary self-embeddings of $A$; in [1], the author proved a equivalence between the Ramsey Property and a somewhat ad hoc two-part version of the Generic Infinitary Ramsey Property, so this article refines that characterization.

Using our result on recovering generic expansions, we develop yet another characterization of the Ramsey Property, this time separating it into a Ramsey Property for 1-element structures (a Pigeonhole Principle) and a “1-simpliciality” property that provides a natural framework for induction arguments for the Ramsey Property in certain classes. To demonstrate the efficacy of this characterization, we provide new proofs of the Ramsey Property for the classes (i) of finite vertex-ordered graphs and (ii) finite trees (equipped with a certain kind of linear order).

0.0.1. Outline. In the remainder of this introductory section, we first provide background, notation, and a general framework for the model theory of classes of finite structures. After that, we provide the basic definitions necessary for discussing Ramsey theorems in a model-theoretic setting, and we review some results proved in [1] and elsewhere.

In the second section, we set up and prove the first main theorem on the existence of $\aleph_0$-categorical generic expansions constrained by an arbitrary expansion. This argument is essentially an exercise in selecting the “right” ultrafilter on the right set associated with
a countable countably-categorical structure. After proving existence, we examine a weakened version of genericity (countable-genericity) that is natural to the Stone space of all of the “right” ultrafilters, and using this, we show that the generic expansions derived from countably-generic ultrafilters are isomorphic. Thus, there is a canonical $\aleph_0$-categorical generic expansion constrained by a given arbitrary expansion.

In the third section, we develop the notion of 1-simpliciality, proving that a class $K$ has the Ramsey Property if and only if it both has the Pigeonhole Principle (the Ramsey Property for its 1-element structures) and is 1-simplicial. Finally, we use this characterization to prove two structural Ramsey theorems that were previously known but whose known proofs were quite complicated and depended heavily on other Ramsey theorems.

0.1. Background on coherent classes and notation.

Definition 0.1 (Eccentric notation for finite structures). In this article, we will usually be working both with classes $K$ of finite structures and infinite models associated with $K$. In order to save ourselves from writing “...is a finite structure” and “is a finite substructure of” ad nauseam, we establish a slightly eccentric convention. With the one exception of $\ell$-embeddings (see the next subsection), upper-case calligraphic letters – $A, B$ and so forth – always denote infinite structures. On the other hand, lower-case gothic letters – $a, b, c$ and so forth – always denote finite structures. Thus, $a \leq A$, for example, means that $a$ is a finite (induced) substructure of $A$. Moreover, given a 1-sorted structure $a$, $|a|$ is its universe (a finite set), and $|a|$ is the cardinality of $a$.

As there doesn’t seem to be a better place to introduce it: we write $\text{Emb}(a, A)$, $\text{Emb}(a, b)$, $\text{Emb}(A, B)$ for the sets of all embeddings $a \to A$, $a \to b$ and $A \to B$, respectively. For an embedding $u$ in one of these sets, $u a$ (or $u A$) is the substructure of the codomain induced on the image of $u$.

Remark 0.2 (Language Convention). In this article, the signature $\text{sig}(\mathcal{L})$ of every language in play will consist of countably many relation symbols. In the case of a coherent class $K$ arising in various hypotheses, we will assume (as we may by classical results on $\aleph_0$-categorical theories in countable languages) that together $K$ and $\mathcal{L}$ satisfy the following:

For all $0 < n < \omega$, $\text{sig}(\mathcal{L})$ has only finite many relation symbols of arity $\leq n$, and for every such relation symbol $R \in \text{sig}(\mathcal{L})$ every $a \in K$, and $\bar{a} \in R^a$, $a_i \neq a_j$ whenever $i < j < n$.

Definition 0.3 (JEP-, Coherent and semi-coherent classes). Consider a class $K$ of finite $\mathcal{L}$-structures (satisfying the language convention) that is closed under isomorphism. First, we express five properties that $K$ could have.

C1. $K$ has the Joint-embedding Property (JEP): for all $a_0, a_1 \in K$, there is a $b \in K$ such that $\text{Emb}(a_0, b) \neq \emptyset$ and $\text{Emb}(a_1, b) \neq \emptyset$.

C2. $K$ has the Amalgamation Property over models (AP): For all $a, b_0, b_1 \in K$ and $f_i \in \text{Emb}(a, b_i)$ ($i < 2$), there are $c \in K$ and $g_i \in \text{Emb}(b_i, c)$ such that $g_0 \circ f_0 = g_1 \circ f_1$.

C3. $K$ has the Heredity Property (HP): For all $b \in K$ and $a \leq b$, $a \in K$.

C4. $K$ has the Weak Löwenheim-Skolem Property (WLS): There is a function $\text{LS} : \omega \to \omega$ such that for all $a \in K$ and $X \subseteq |a|$, there are $b, c \in K$ such that $a \leq c$, $b \leq c$, $X \subseteq |b|$, and $|b| \leq \text{LS}(|X|)$.
C5. $K$ has the Strong Amalgamation Property (SAP) over models: For all $a, b_0, b_1 \in K$ and $f_i \in \text{Emb}(a, b_i)$ ($i < 2$), there are $c \in K$ and $g_i \in \text{Emb}(b_i, c)$ such that $g_0 \circ f_0 = g_1 \circ f_1$ and

$$g_0 b_0 \cap g_1 b_1 = g_0 f_0 a = g_1 f_1 a.$$ 

Now, we make a few definitions of different types of classes (with some examples).

- $K$ is a JEP-class if it has $C_1 (=\text{JEP})$.
- $K$ is a semi-coherent class if it has $C_1$ and $C_2$ (JEP and AP).
  
  Example: For a prime $p$, the class of all finite fields of characteristic $p$.
- $K$ is a coherent class if it has $C_1$, $C_2$, and $C_4$.
  
  Example: In a relational language, for a finite field $\mathbb{F}$, the class of all finite $\mathbb{F}$-vector spaces.
- $K$ is a Fraïssé class if it has $C_1$, $C_2$, and $C_3$.
  
  Example: The class of all finite graphs.

It is not very difficult to verify (using the given examples), that these types of classes are related as follows:

$$\{\text{JEP-classes}\} \supset \{\text{semi-coherent classes}\} \supset \{\text{coherent classes}\} \supset \{\text{Fraïssé classes}\}$$

Remark 0.4. While we discuss generic models in detail in the next section, we need this notion at least in a loose sense for some formulations about Ramsey theory. So, suppose $K$ is a semi-coherent class of finite $L$-structures. A countably infinite structure $A$ is generic for $K$ if it satisfies the following criteria:

- Universality: For all $a \in K$, $\text{Emb}(a, A) \neq \emptyset$.
- Homogeneity: For all $a, b \in K$ with $a \leq b$ and all $f_0 \in \text{Emb}(a, A)$, there is an $f \in \text{Emb}(b, A)$ extending $f$ (i.e. $f_0 \subseteq f$ for logicians).

It can be shown (and will be shown implicitly in the next section) that up to isomorphism, a semi-coherent class $K$ has exactly one generic model, so the theory $T_K := \text{Th}(A)$ is well-defined. We call $T_K$ the generic theory of $K$. The random graph and the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ are examples of generic models of the class of finite graphs and finite fields of characteristic $p$, respectively.

However, $T_K$ may have additional countable models; for example, $\overline{\mathbb{F}}_p$ is itself has no transcendental elements, but $\text{Th}(\overline{\mathbb{F}}_p)$ has countable models that do. That is to say, $T_K$ need not be $\aleph_0$-categorical. A result of [3] holds that for a semi-coherent class $K$, $T_K$ is $\aleph_0$-categorical if and only if $K$ is coherent.

In our discussion of generic expansions, we will need one more type of class of finite structures and one notion of equivalence for classes of finite structures. The first concept is a weakening/generalization of semi-coherence that is known as the almost-amalgamation property in [2].

**Definition 0.5** (guardedly coherent classes). Let $K$ be a JEP class. We say that $K$ is guardedly-coherent if for every $a \in K$, there is an $a \leq a_1 \in K$ such that for all $b_0, b_1 \in K$ and $f_i \in \text{Emb}(a_1, b_i)$ ($i < 2$), there are $c \in K$ and $g_i \in \text{Emb}(b_i, c)$ ($i < 2$) such that $(g_0 \circ f_0)|a = (g_1 \circ f_1)|a$. 


Definition 0.6. Suppose $K_1, K_2$ are JEP-classes in the same language $\mathcal{L}$. We say that $K_1$ and $K_2$ are cofinal (or $K_1$ is cofinal with $K_2$, and so forth) if:

- For all $a \in K_1$, there is a $b \in K_2$ such that $a \leq b$.
- For all $a \in K_2$, there is a $b \in K_1$ such that $a \leq b$.

We note that if $K_1, K_2$ are both semi-coherent, saying $K_1$ and $K_2$ are cofinal is equivalent to saying that they have the same generic theory.

To almost-conclude this subsection on background and notation, we make one more set of definitions that are very convenient later on, but not in any way deep or essential. To truly conclude this subsection, we make a final remark about the generic model of a coherent class with SAP.

Definition 0.7. Let $K$ be a semi-coherent class, and let $A$ be its generic model.

- In analogy with the older notion of the age of a structure, we define
  
  $K(A) = \{a \in K : a \leq A\}$.

  Indeed, if $K$ happens to be a Fraïssé class, then $K(A)$ is precisely the age of $A$.

- For $a \in K$, we define
  
  $K_a = \{b \in K : a \leq b\}$

  and if $a \in K(A)$,

  $K_a(A) = \{b \in K(A) : a \leq b\}$.

  The latter we might call the cone above $a$ in $A$.

- Finally, we define
  
  $\text{Cone}(A) = \{X \subseteq K(A) : (\exists a \in K(A)) K_a(A) \subseteq X\}$.

  It is not difficult to verify that $\text{Cone}(A)$ is a proper filter on $K(A)$.

Remark 0.8. Let $K$ be a coherent class with generic model $A$; the following are equivalent:

1. $K$ has SAP.
2. For every $a \in K(A)$, $\text{acl}^A(\|a\|) = \|a\|$.

Usually, we will economize in type-setting by write $a \cap b = c$ in place of $\|a\| \cap \|b\| = \|c\|$, and so forth. Thus, SAP is equivalent to the condition that $\text{acl}^A(a) = a$ for all $a \in K(A)$.

We also note that, regardless of SAP, for any $a_0 \in K(A)$ and $a_1 \in K$, there is an embedding $f \in \text{Emb}(a_1, A)$ such that $a_0 \cap fa_1 = \text{acl}^A(\emptyset)$.

0.2. Background on structural Ramsey theory. The original motivation for this article lies in structural Ramsey theory, so before we begin presenting new results, we recall the necessary basic material on structural Ramsey theory in the context of coherent classes. Our formulation here is based on the development in the present author’s work [1] with minor and inessential modifications.

The original form of the Ramsey Property – or the Finitary Ramsey Property – is the following:

Definition 0.9 (Ramsey Property). Let $K$ be a coherent class.

- For $a \in K$, we say that $K$ has the $a$-Ramsey Property ($a$-RP) if for all $0 < k < \omega$ and all $b \in K$, there is a $c \in K$ such that for every coloring $\xi : \text{Emb}(a, c) \to k$, there is an embedding $u \in \text{Emb}(b, c)$ such that $\xi$ is constant on $\text{Emb}(a, ub)$ – that is, $\xi(u \circ f) = \xi(u \circ f')$ for all $f, f' \in \text{Emb}(a, b)$.
• We say that \( K \) has the Ramsey Property (RP) if \( K \) has the \( a \)-Ramsey Property for every \( a \in K \).

The transfer from the classical Ramsey Property to what was called the Amalgamated Ramsey Property is not deep, but it is extremely convenient for other demonstrations. Essentially, it is just a weak method of “pulling up” the Ramsey Property for a coherent class \( K \) to a property of its generic model; we call it “weak” because the notion of an \( \ell \)-embedding is rather \textit{ad hoc}, so that ARP calls out to be replaced by a property expressed entirely in terms of elementary embeddings of the generic model.

\textbf{Definition 0.10 (Amalgamated Ramsey Property).} Let \( K \) be a coherent class with countable generic model \( \mathcal{A} \).

• An \( \ell \)-embedding \( u : \mathcal{A} \hookrightarrow \mathcal{A} \) is really a family of embeddings \( u_X \in \text{Emb}(\mathcal{A}_X, \mathcal{A}) \), \( X \subset_{\text{fin}} \mathcal{A} \), together with an implicit bounding function \( s : \omega \to \omega \) such that for every \( X \subset_{\text{fin}} \mathcal{A} \), the following hold:
  - \( \mathcal{A}_X \in K(\mathcal{A}) \) and \( X \subseteq ||\mathcal{A}_X|| \);
  - \( |\mathcal{A}_X| \leq s(|X|) \).

• For \( a \in K \), we say that \( K \) has the \( a \)-Amalgamated Ramsey Property (\( a \)-ARP) if for all \( 0 < k < \omega \), for every coloring \( \xi : \text{Emb}(a, \mathcal{A}) \to k \), there is an \( \ell \)-embedding \( u : \mathcal{A} \hookrightarrow \mathcal{A} \) such that for all \( X \subset_{\text{fin}} \mathcal{A} \), \( \xi \) is constant on \( \text{Emb}(a, h_X[\mathcal{A}_X]) \) – that is to say, \( \xi(u_X \circ f) = \xi(u_X \circ f') \) for all \( f, f' \in \text{Emb}(a, \mathcal{A}_X) \).

• We say that \( K \) has the Amalgamated Ramsey Property (ARP) if \( K \) has \( a \)-ARP for every \( a \in K \).

The next definition is the first step towards replacing \( \ell \)-embeddings with genuine embeddings in a formulation of the Ramsey Property. Several statements herein appeal to definitions in the next section (on generic expansions in general)

\textbf{Definition 0.11 (Coloring expansions).} Let \( \mathcal{L} \) be a finitely-rigid\(^1\) coherent class in \( \mathcal{L} \), and let \( a \in K \) and \( 0 < k < \omega \). We define \( \mathcal{L}_{a,k} \) to be the expansion of \( \mathcal{L} \) by \( k \) new relation symbols \( R_0, \ldots, R_{k-1} \), all of arity \( r = |a| \).

• For \( b \in K \) and a coloring \( \xi : \text{Emb}(a, b) \to k \), we define the \( \mathcal{L}_{a,k} \)-expansion \( b^\xi \) of \( b \) by interpreting

\[
R_i^{b^\xi} = \{ fa : f \in \text{Emb}(a, b), \xi(f) = i \}
\]

for each \( i < k \). If \( \mathcal{A} \) is a generic model for \( K \) and \( \xi : \text{Emb}(a, \mathcal{A}) \to k \) is a coloring, then we define an \( \mathcal{L}_{a,k} \)-expansion \( \mathcal{A}^\xi \) of \( \mathcal{A} \) in the same way.

• For a coloring \( \xi : \text{Emb}(a, \mathcal{A}) \to k \), we define...

... \( K^\xi \) to be the class of \( \mathcal{L}_{a,k} \)-structures obtained by closing the set of structures \( \{ b^\xi b : b \in K(\mathcal{A}) \} \) under isomorphisms.

... \( K^\xi(\mathcal{A}) \) to be the set \( \{ b^+ \in K^\xi : b^+\mathcal{L} \subseteq K(\mathcal{A}) \} \).

... \( X_\xi \) to be the set of \( \mathcal{L}_{a,k} \)-expansions \( \mathcal{A}^\xi \) of \( \mathcal{A} \) such that for every \( b \in K(\mathcal{A}) \), \( \mathcal{A}^\xi \upharpoonright ||b|| \in K^\xi(\mathcal{A}) \).

Finally, we say that \( \xi \) is a \textit{generic coloring} if \( \mathcal{A}^\xi \) is \( \aleph_0 \)-categorical and its \( \text{Aut}(\mathcal{A}) \)-orbit in \( X_\xi \) is comeagre.

\(^1\)Meaning that for every \( a \in K \), \( \text{Aut}(a) \) is trivial.
• Suppose \( \xi_0 : \text{Emb}(a, \mathcal{A}) \to k \) is an arbitrary coloring, and let \( \mathcal{A}^+ \) be the \( \aleph_0 \)-categorical generic expansion of \( \mathcal{A} \) constrained by \( \mathcal{A}^\xi_0 \); we define \( \xi : \text{Emb}(a, \mathcal{A}) \to k \) by

\[
\xi(f) = i \iff f a \in R_i^{i+}
\]

for each \( i < k \). Then it is easily seen that \( \xi \) is a generic coloring, which we call the generic coloring constrained by \( \xi \).

In [1], we took a small step towards characterizing the Ramsey Property for \( K \) in terms of self-embeddings of the generic model. The proof in [1] amounted to demonstrating that \( \mathfrak{a} \text{-ARP} \iff \mathfrak{a} \text{-WGIRP} \), and as it seems relevant to the development in this article, we give a sketch of the proof of this equivalence below (simplified using technology developed later in this paper).

**Theorem 0.12 (form [1]).** Let \( K \) be a coherent class with countable generic model \( \mathcal{A} \), and let \( a \in K \). The following are equivalent:

1. \( K \) has the \( \mathfrak{a} \)-Ramsey Property.
2. \( K \) has the \( \mathfrak{a} \)-Amalgamated Ramsey Property.
3. \( K \) has the \( \mathfrak{a} \)-Weak Generic Infinitary Ramsey Property (\( \mathfrak{a} \text{-WGIRP} \)) in two parts:
   - For all \( 0 < k < \omega \), for any generic coloring \( \xi : \text{Emb}(a, \mathcal{A}) \to k \), there is an elementary embedding \( u : \mathcal{A} \to \mathcal{A} \) such that \( \xi \) is constant on \( \text{Emb}(a, u \mathcal{A}) \) — that is, \( \xi(u \circ f) = \xi(u \circ f') \) for all \( f, f' \in \text{Emb}(a, \mathcal{A}) \).
   - For all \( 0 < k < \omega \), for any coloring \( \xi : \text{Emb}(a, \mathcal{A}) \to k \), there is a generic coloring \( \xi_0 : \text{Emb}(a, \mathcal{A}) \to k \) constrained by \( \xi \).

**Proof (sketch) of \( \mathfrak{a} \text{-ARP} \Rightarrow \mathfrak{a} \text{-WGIRP} \).** Given an arbitrary coloring \( \xi_0 : \text{Emb}(a, \mathcal{A}) \to k \), we pass to a generic coloring \( \xi : \text{Emb}(a, \mathcal{A}) \to k \) constrained by \( \xi_0 \) using the results of the next section. Let \( (b_n)_{n<\omega} \) be a chain of members of \( K \) such that \( b_n \leq b_{n+1} \leq \mathcal{A} \) for all \( n < \omega \) and \( \bigcup_n b_n = \mathcal{A} \). For each \( n < \omega \), let \( \Gamma_n \) be the set of all embeddings \( w : b_n \to \mathcal{A} \) such that \( \xi \) is constant on \( \text{Emb}(a, w b_n) \). Further, for each \( n < \omega \), let \( \sim_n \) be the equivalence relation on \( \Gamma_n \) given as follows: \( w \sim_n w' \) if there is an automorphism \( g \in \text{Aut}(\mathcal{A}^\xi) \) such that \( w' b_n = g w b_n \). Now, \( \Gamma = \bigcup_n \Gamma_n \) is a tree (under \( \subseteq \)), for if \( m \leq n \) and \( w \in \Gamma_n \), then \( w' b_m \in \Gamma_m \). Defining \( \Gamma_n^* = \Gamma_n / \sim_n \) for each \( n < \omega \), we can adjust the partial-order relation \( \subseteq \) so that \( \Gamma^* = \bigcup_n \Gamma_n^* \) is a finitely-branching tree. If \( \Gamma^* \) is infinite, then by König’s Lemma, it has infinite branch \( (w_n / \sim_n \in \Gamma_n^*)_{n<\omega} \), and using the genericity of \( \xi \), one can easily-enough convert the \( (w_n / \sim_n)_{n<\omega} \) into an elementary embedding \( w : \mathcal{A} \to \mathcal{A} \) such that \( w \upharpoonright b_n \sim_n w_n \) for every \( n < \omega \). In particular, \( \xi \) is constant on \( \text{Emb}(a, w b_n) \). Thus, to complete the demonstration of \( \mathfrak{a} \text{-WGIRP} \), it is enough to verify that \( \Gamma^* \) is indeed infinite, and for this it is sufficient to verify that each \( \Gamma_n \) is non-empty. And finally, by \( \mathfrak{a} \text{-ARP} \) applied to \( \xi \), for each \( n < \omega \), there is an embedding \( v : b_n \to \mathcal{A} \) such that \( \xi \) is constant on \( \text{Emb}(a, v b_n) \); then, \( v \in \Gamma_n \), and the proof is completed.

**Proof (sketch) of \( \mathfrak{a} \text{-WGIRP} \Rightarrow \mathfrak{a} \text{-ARP} \).** Given an arbitrary coloring \( \xi_0 : \text{Emb}(a, \mathcal{A}) \to k \), we pass to a generic coloring \( \xi : \text{Emb}(a, \mathcal{A}') \to k \) constrained by \( \xi_0 \) — in this case, by \( \mathfrak{a} \text{-WGIRP} \) we suppose — where for clarity, we take \( \mathcal{A}' \) to be a disjoint isomorphic copy of \( \mathcal{A} \). Then, for any \( b \leq \mathcal{A} \), there is an embedding \( v : b \to \mathcal{A} \) such that \( \xi(f) = \xi_0(v \circ f) \) for every \( f \in \text{Emb}(a, b) \). Again by \( \mathfrak{a} \text{-WGIRP} \), we recover an elementary embedding \( w : \mathcal{A} \to \mathcal{A} \) such that \( \xi \) is constant on \( \text{Emb}(a, w \mathcal{A}) \).
To define the required ℓ-embedding \( u : \mathcal{A} \rightarrow \mathcal{A} \), let \( X \subset \text{fin} \mathcal{A} \) be given. Let \( A_X \leq A \) be a finite substructure of \( A \) containing \( X \) such that \( |A_X| \leq LS(|X|) \); let \( v_X : A_X \rightarrow A \) be any embedding whatever, and let \( v'_X : v_X A_X \rightarrow w A \) be any embedding at all – so that \( \xi(f) = \xi(v'_X \circ f) \) for all \( f \in \text{Emb}(a, v_X A_X) \). Taking \( u_X = v'_X \circ v_X \), it follows that \( \xi_0 \) is constant on \( \text{Emb}(a, u_X A_X) \), as required. □

Removing the “existence of generic colorings” condition in the formulation of the \( a \)-Weak Generic Infinitary Ramsey Property is the motivating problem of this article. Thus, a primary goal for us is to prove the following refinement of the previous theorem. Once we establish this – by showing that the “existence of generic colorings” condition is simply always true for coherent classes – we will have established a more natural characterization of the Ramsey Property in terms of (elementary) embeddings of the the generic model; this is expressed in Corollary 0.14.

**Theorem 0.13.** Let \( K \) be a coherent class with SAP and with countable generic model \( A \), and let \( a \in K \). The following are equivalent:

1. \( K \) has the \( a \)-Ramsey Property.
2. \( K \) has the \( a \)-Amalgamated Ramsey Property.
3. \( K \) has the \( a \)-Generic Infinitary Ramsey Property (\( a \)-GIRP) in one part:
   For all \( 0 < k < \omega \), for any generic coloring \( \xi : \text{Emb}(a, A) \rightarrow k \), there is an elementary embedding \( u : A \rightarrow A \) such that \( \xi \) is constant on \( \text{Emb}(a, u A) \).

**Corollary 0.14.** Let \( K \) be a coherent class with SAP and with countable generic model \( A \), and let \( a \in K \). The following are equivalent:

1. \( K \) has the Ramsey Property.
2. \( K \) has the Amalgamated Ramsey Property.
3. \( K \) has the Generic Infinitary Ramsey Property – meaning, of course, that \( K \) has \( a \)-GIRP for all \( a \in K \).

1. **Constrained generic expansions of \( \aleph_0 \)-categorical structures**

   Throughout this section, \( K \) is a coherent class, and \( A \models T_K \) is its countable generic model. Although we are specifically interested in color expansions of \( A \), we formulate and prove our results in terms of nearly-arbitrary expansions of \( A \). This allows us build more or less directly upon the development in [2].

   In the first subsection below, we recall several of the relevant definitions from [2], leading up to the statement of our first main theorem, Theorem 1.6. In the subsequent subsection, we prove this theorem.

1.1. **Background on generic expansions of \( \aleph_0 \)-categorical structures.** In the first definitions to follow, we establish the basic context for discussing expansions of \( \aleph_0 \)-categorical structures. Under slightly different names, all of these ideas appeared previously in [2].

**Definition 1.1** (\( T_K \)-normal sentences). Let \( \mathcal{L}^+ \) be an expansion of \( \mathcal{L} \) by finitely many additional relation symbols. For our purposes, a \( T_K \)-normal sentence of \( \mathcal{L}^+ \) is an \( \mathcal{L}^+ \)-sentence of the form,

\[
\forall x_0 \ldots x_{n-1} (\theta_0(\bar{x}) \rightarrow \theta(\bar{x}))
\]

where \( \theta_0 \) isolates a complete \( n \)-type of \( T_K \) and \( \theta \) is a boolean combination of formulas of \( \mathcal{L} \) and quantifier-free formulas of \( \mathcal{L}^+ \).
Definition 1.2 \((K^\Sigma \text{ and } K^\Sigma(A); \mathbb{X}_\Sigma)\). Let \(\Sigma\) be a set of \(T_K\)-normal sentences such that \(T_K \cup \Sigma\) is consistent.

- We define \(K^\Sigma\) to be the set of finite structures \(\mathcal{B}|X\), where \(\mathcal{B} \models T_K \cup \Sigma\), \(X <_{\text{fin}} \mathcal{B}\) and the reduct \((\mathcal{B}|X)|\mathcal{L}\) is in \(K\). Subsequently, we define \(K^\Sigma(A)\) to be the set of \(\mathcal{L}^+\)-structures \(a^+ \in K^\Sigma\) such that \(a^+|\mathcal{L} \in K(A)\).
- We define \(\mathbb{X}_\Sigma\) to be the set of all \(\mathcal{L}^+\)-expansions \(A'\) of \(\mathcal{A}\) such that \(\mathcal{A}' \models T_K \cup \Sigma\). We topologize \(\mathbb{X}_\Sigma\) by taking
  \[
  \{[a^+] : a^+ \in K^\Sigma(A)\}
  \]
  as a base of (cl)open sets, where for each \(a^+ \in K^\Sigma(A)\),
  \[
  [a^+] = \{A' \in \mathbb{X}_\Sigma : a^+ \leq_{\mathcal{L}^+} A'\}.
  \]

We also note that this topology on \(\mathbb{X}_\Sigma\) is compactly metrizable, so \(\mathbb{X}_\Sigma\) is a Baire space. (In fact, \(\mathbb{X}_\Sigma\) is a Cantor space.)

The first main theorem of [2] is the following, completely characterizing when a given set of \(T_K\)-normal sentences supports a generic expansion of a countably-categorical structure. Although it was proved earlier, the result of [6] can be obtained easily from this theorem and the theorem of [3] noted above.

**Theorem 1.3** ([2]). Let \(\Sigma\) be a set of \(T_K\)-normal sentences. The following are equivalent:

1. \(\mathbb{X}_\Sigma\) has a comeagre \(\text{Aut}(\mathcal{A})\)-orbit.
2. \(K^\Sigma\) has a cofinal guarded-coherent subclass.

**Corollary 1.4** ([6]). Let \(\Sigma\) be a set of \(T_K\)-normal sentences. If \(K^\Sigma\) has a coherent cofinal subclass, then \(\mathbb{X}_\Sigma\) has a comeagre \(\text{Aut}(\mathcal{A})\)-orbit, say \(\{\mathcal{A}^+\}^{\text{Aut}(\mathcal{A})}\), and \(\text{Th}(\mathcal{A}^+)\) is \(\aleph_0\)-categorical.

With these basic theorems in place, we now specify what is meant by a “generic expansion of \(\mathcal{A}\) constrained by another structure \(\mathcal{B}\).”

**Definition 1.5** \((\Sigma(\mathcal{B}); \text{constrained generics over } \mathcal{A})\). Let \(\mathcal{L}^+\) be an expansion of \(\mathcal{L}\) by finitely many additional relation symbols, and let \(\mathcal{B}\) be a countable \(\mathcal{L}^+\)-structure such that \(\mathcal{B}|\mathcal{L} \models T_K\).![](cid:0)

We define \(\Sigma(\mathcal{B})\) to be the set of \(T_K\)-normal sentences \(\varphi\) of \(\mathcal{L}^+\) such that \(\mathcal{B} \models \varphi\); clearly, \(\mathcal{B} \models T_K \cup \Sigma(\mathcal{B})\).

Now, let \(\Sigma\) be a set of \(T_K\)-normal sentences such that \(T_K \cup \Sigma\) is consistent and \(\Sigma(\mathcal{B}) \subseteq \Sigma\). Let \(\{\mathcal{A}^+\}^{\text{Aut}(\mathcal{A})}\) be a comeagre \(\text{Aut}(\mathcal{A})\)-orbit of \(\mathbb{X}_\Sigma\); then, \(\mathcal{A}^+\) is a generic expansion of \(\mathcal{A}\) constrained by \(\mathcal{B}\). If, further, \(\text{Th}(\mathcal{A}^+)\) is \(\aleph_0\)-categorical, then we say that \(\mathcal{A}^+\) is an \(\aleph_0\)-categorical generic expansion of \(\mathcal{A}\) constrained by \(\mathcal{B}\).

At last, we can formulate the first main theorem of this paper:

**Theorem 1.6.** Let \(K\) be a coherent class in \(\mathcal{L}\) with SAP, and let \(\mathcal{A}\) be its countable generic model. Let \(\mathcal{L}^+\) be an expansion of \(\mathcal{L}\) by finitely many additional relation symbols, and let \(\mathcal{B}\) be a countable \(\mathcal{L}^+\)-structure such that \(\mathcal{B}|\mathcal{L} \models T_K\). Then there is an \(\aleph_0\)-categorical generic expansion of \(\mathcal{A}\) constrained by \(\mathcal{B}\).
1.2. **Proof of Theorem 1.6.** The difficulty of proving Theorem 1.6 lies entirely in expanding the original set $\Sigma(B)$ of $T_K$-normal sentences to a set $\Sigma$ of $T_K$-normal sentences such that $K^{\Sigma}$ is coherent. Defining such an expansion involves making possibly infinitely-many choices about what “configurations” consistent with $\Sigma(B)$ to keep or discard. Instead of explicitly making these decisions, we instead build an ultrafilter $\Psi$ extending $\text{Cone}(\mathcal{A})$ that will make all of these decisions for us. As is, we suppose, most common in constructing ultrafilters, we don’t explicitly specify $\Psi$ — we use the compactness of the Stone space of ultrafilters extending $\text{Cone}(\mathcal{A})$ with further properties. We must remark, here, that in essence our argument is a forcing construction, but the language of forcing does not seem to us to be especially helpful here.

For the rest of this subsection (that is, the proof of Theorem 1.6), we fix a coherent class $K$ of finite $\mathcal{L}$-structures with SAP, and we take $\mathcal{A}$ to be its generic model. We fix an expansion $\mathcal{L}^+$ of $\mathcal{L}$ by finitely many new relation symbols and a countable $\mathcal{L}^+$-structure $\mathcal{B}$ such that $\mathcal{B} \vDash T_K$. Since $T_K$ is $\aleph_0$-categorical, we assume without loss of generality that $\mathcal{B} \subseteq \mathcal{A}$.

Subsequently, we define $K^B(\mathcal{A}) = \{\mathcal{B} \upharpoonright a : a \in K(\mathcal{A})\}$ and for $a \in K^B(\mathcal{A})$,

$$K^B_a(\mathcal{A}) = \{b \in K^B(\mathcal{A}) : a \leq b\}.$$ 

**Definition 1.7.** For $a \in K^B(\mathcal{A})$, we define several subsets of $K(\mathcal{A})$:

- $U_a$ is the set of all $b \in K^B_a(\mathcal{A})$ such that for every finite set $\|a\| \subseteq X \subseteq \mathcal{A}$, there is an automorphism $g \in Aut(\mathcal{A}/a)$ such that $gb \cap X = a$ and the induced map $b \to gb$ is an isomorphism of $\mathcal{L}^+$-structures.
- $V_a = \bigcup \{U_g : g \in Aut(\mathcal{A}) \text{ and } a \to g a \text{ is an isom. of } \mathcal{L}^+\text{-struct.'s}\}.$

We note that for each $a_0 \in K(\mathcal{A})$,

$$V_{f_{a_0}} : f \in \text{Emb}(a_0, \mathcal{A})$$

is a finite set.

For $a_0 \in K(\mathcal{A})$, we define $W_{a_0} = \bigcup_{f \in \text{Emb}(a_0, \mathcal{A})} V_{f_{a_0}}$.

Towards proving Theorem 1.6, the key step is the following Proposition 1.8. By basic facts about filters (and set families with the finite intersection property), the proof of this Proposition 1.8 boils down entirely to Lemma 1.9 below.

**Proposition 1.8.** The family

$$\text{Cone}^B(\mathcal{A}) = \left\{ X \subseteq K(\mathcal{A}) : (\exists 0 < n < \omega, a_0, a_1, \ldots, a_n \in K(\mathcal{A})) K_{a_n}(\mathcal{A}) \cap \bigcap_{i < n} W_{a_i} \subseteq X \right\}$$

is a proper filter on $K(\mathcal{A})$.

**Lemma 1.9.** Let $0 < n < \omega$ and $a_0, a_1, \ldots, a_n \in K(\mathcal{A})$. Then $K_{a_n}(\mathcal{A}) \cap \bigcap_{i < n} W_{a_i}$ is non-empty. (In fact, it is infinite.)
**Proof.** Let $0 < n < \omega$ and $a_0, a_1, \ldots, a_n \in K(\mathcal{A})$ be given. By the SAP assumption, we may assume, without loss of generality, that $acl^\mathcal{A}(\bigcup_{i<n} a_i) \cap a_n = acl^\mathcal{A}(\emptyset)$ and $a_i \cap a_j = acl^\mathcal{A}(\emptyset)$ for all $i < j \leq n$. Let $X_0 \subset \cdots X_i \subset \cdots \subset \text{fin } A$ be a chain of finite subsets of $A$ such that $\bigcup_i X_i = A$ and $\|a_n\| \subseteq X_0$. Let $c \in \bigcap_{i \leq n} K_{a_i}(\mathcal{A})$ be such that $\|c\| \supseteq acl^\mathcal{A}(\bigcup_{i \leq n} a_i)$. For each $i < \omega$, we may choose inductively $g_i \in Aut(\mathcal{A}/a_n)$ such that

$$g_i c \cap \left( X_i \cup \bigcup_{j<i} g_j c_j \right) = a_n.$$ 

Let $Q_\xi$ be the set $\{ (\mathcal{B}||c||)/\cong_{a_n} : e \in \text{Emb}(c, \mathcal{A}), a_n \subseteq e \}$ \textendash a finite set \textendash and consider the function $\xi : \omega \rightarrow Q_\xi$ given by

$$\xi(i) = (\mathcal{B}||g_i c||)/\cong_{a_n}.$$ 

Since $Q_\xi$ is finite, by the Pigeonhole Principle, there is an infinite subset $I \subseteq \omega$ on which $\xi$ is constant. Then

$$\{ g_i c : i \in I \} \subseteq K_{a_n}(\mathcal{A}) \cap \bigcap_{i<n} W_{a_i}$$

as required. \hfill \Box

The final building-block in the proof of Theorem 1.6 is the following observation (whose proof is basically trivial, so we omit it).

**Observation 1.10.** Let $\Psi$ be a non-principal ultrafilter on $K(\mathcal{A})$ extending $\text{Cone}^\mathcal{B}(\mathcal{A})$, and define

$$K^\Psi(\mathcal{A}) = \{ a \in K^\mathcal{B}(\mathcal{A}) : V_a \in \Psi \}.$$ 

Also, let $K^\Psi$ be the closure of $K^\Psi(\mathcal{A})$ under isomorphisms. Then $K^\Psi$ is a coherent class.

**Proof of Theorem 1.6.** Let $K$ be a coherent class in $\mathcal{L}$ with SAP, and let $\mathcal{A}$ be its countable generic model. Let $\mathcal{L}^+$ be an expansion of $\mathcal{L}$ by finitely many additional relation symbols, and let $\mathcal{B}$ be a countable $\mathcal{L}^+$-structure such that $\mathcal{B} \models T_K$. Let $\Psi$ be an ultrafilter extending $\text{Cone}^\mathcal{B}(\mathcal{A})$ as in the previous observation, so that $K^\Psi$ is coherent; it is easily verified that $K^\Psi$ is a cofinal subclass of $K^{\Sigma(\mathcal{B})}$. By Corollary 1.4 and the fact that $K^\Psi$ is coherent, $X_\Sigma$ has a comeagre $Aut(\mathcal{A})$-orbit $\{ A^+ \}_{\text{Aut}(\mathcal{A})}$ such that $Th(A^+)$ is $\aleph_0$-categorical, where $\Sigma$ is the set of $T_K$-normal sentences derived from $K^\Psi$. Then, $\mathcal{A}^+$ is an $\aleph_0$-categorical generic expansion of $\mathcal{A}$ constrained by $\mathcal{B}$, as desired. \hfill \Box

1.3. **Some additional remarks on constrained generic expansions.** For this subsection, as in the last, we fix a coherent class $K$ of finite $\mathcal{L}$-structures with SAP, and we take $\mathcal{A}$ to be its generic model. Again, we fix an expansion $\mathcal{L}^+$ of $\mathcal{L}$ by finitely many new relation symbols and a countable $\mathcal{L}^+$-structure $\mathcal{B}$ such that $\mathcal{B} \models T_K$. And again, since $T_K$ is $\aleph_0$-categorical, we assume that $\mathcal{B} \models \mathcal{L} = \mathcal{A}$, and we define

$$K^\mathcal{B}(\mathcal{A}) = \{ \mathcal{B} || a || : a \in K(\mathcal{A}) \}$$

and for $a \in K^\mathcal{B}(\mathcal{A})$,

$$K^a(\mathcal{A}) = \{ b \in K^\mathcal{B}(\mathcal{A}) : a \leq b \}.$$ 

We maintain the notations $U_a$, $V_a$ and $W_a$ as in the previous subsection as well.
Remark 1.11 (Countably-generic ultrafilter). Let \( Z \) be some infinite set, and let \( C \) be a closed subset of the Stone space \( \beta Z \) of all ultrafilters on \( Z \). As usual, \( \beta Z \) is topologized with a base of clopen sets of the form \([X] = \{ p \in \beta Z : X \subseteq p \}\), where \( X \subseteq Z \). Let \( W \subseteq \omega^{<\omega} \) be a tree with no finite maximal chains, and let \( \pi : W \to \mathcal{P}(Z) \); we shall call \( \pi \) a probe on \( \beta Z \) if it satisfies the following three conditions:

- \( \pi(\sigma) \) is infinite for every \( \sigma \in W \);
- \( \pi(\sigma) \supseteq \pi(\tau) \) whenever \( \sigma \subseteq \tau \in W \);
- \( \pi(\sigma) \cap \pi(\tau) = \emptyset \) whenever \( \sigma, \tau \in W \) are incomparable.

Defining \( M(W) \) to be the set of all maximal chains in \( W \), we define

\[
S_\pi = \bigcup \left\{ \bigcap [\pi(f)[k] : f \in M(W), k < \omega \} \right. 
\]

Then, we define \( I(C) \) to be the \( \sigma \)-ideal of subsets of \( C \) generated by,

\[
\{ S_\pi \cap C : \pi \text{ is a probe on } \beta Z \} .
\]

(That is, we iteratively close the latter under subset and countable unions.) For lack of better terminology, we will say that a countably-generic element of \( C \) is an ultrafilter \( p \in C \setminus \bigcup I(C) \). When \( C \) is clear from context, we will just say that \( p \in C \setminus \bigcup I(C) \) is countably-generic.

Now, we observe that \( C \setminus \bigcup I(C) = C \setminus \bigcup S_\pi \) where \( \pi \) ranges over all probes. Moreover, \( C \) being a closed subset of a compact space \( \beta Z \) – hence, compact itself – to show that \( C \setminus \bigcup S_\pi \) is non-empty, it suffices to show that for any finite family of probes \( \pi_0, ..., \pi_n \), the intersection \( \bigcap_{i<n} (C \setminus S_{\pi_i}) \) is non-empty. Thus, \( C \) has a countably-generic element if and only if the family

\[
\Pr(C) = \{ C \setminus S_\pi : \pi \text{ is a probe} \} 
\]

has the finite intersection property – i.e. for any finite family \( F \subseteq \Pr(C), \bigcap F \neq \emptyset \).

Definition 1.12. Let \( \mathcal{E}^B(A) \) be the set of all ultrafilters \( \Psi \) on \( K(A) \) such that \( \text{Cone}^B(A) \subseteq \Psi \). We note that \( \mathcal{E}^B(A) \) is a closed subset of \( \beta K(A) \), for

\[
\mathcal{E}^B(A) = \bigcap \{ [X] : X \in \text{Cone}^B(A) \}
\]

where for each \( X \subseteq K(A), [X] = \{ \Psi \in \beta K(A) : X \subseteq \Psi \}. \) (Since each \([X] \) is clopen, \( \mathcal{E}^B(A) \) is also a \( G_\delta \)-subset of \( \beta K(A) \).) It is more or less routine to verify that \( \Pr(\mathcal{E}^B(A)) \) has the finite intersection property, so \( \mathcal{E}^B(A) \) does have a countably-generic element.

Proposition 1.13. Let \( \Psi_0, \Psi_1 \in \mathcal{E}^B(A) \). If \( \Psi_1 \) is countably-generic, then \( K^{\Psi_0} \subseteq K^{\Psi_1} \).

Proof. Let \( \Psi_1 \) be a countably-generic element of \( \mathcal{E}^B(A) \). For each \( a \in K(A) \), we observe that either \( K_0(a) \setminus V_a = \emptyset \) for some \( b \in K(A) \) (in which case \( B \upharpoonright \|a\| \) is not in any \( K^{\Psi}, \Psi \in \mathcal{E}^B(A) \)) or we may define (non-canonically) a probe \( \pi \) as follows:

- \( \pi(\emptyset) \) is an infinite subset of \( K(A) \setminus V_a \).
- Given \( \sigma \in \omega^{<\omega} \) and \( \pi(\sigma) \) already defined and infinite, we select any partition \( \{X_i\}_{i<\omega} \) of \( \pi(\sigma) \) into infinite sets, and we define \( \pi(\sigma^\frown i) = X_{i+1} \) for each \( i < \omega \).

Let \( \Pr(a) \) be the set of all probes defined in this manner. Clearly, \( \Pr(a) \subseteq \Pr(\mathcal{E}^B(A)) \), implying that

\[
\Psi_1 \in \mathcal{E}^B(A) \setminus \bigcup_{\pi \in \Pr(a)} S_\pi .
\]
and it follows that $B\parallel a \in K^{\Psi_1}$. As $a \in K(A)$ was arbitrary, we have shown that for any $\Psi_0 \in C^B(A)$ and any $a \in K(A)$, if $B\parallel a \in K^{\Psi_0}$, then $B\parallel a \in K^{\Psi_1}$ as well, which completes the proof the proposition.

\[ \square \]

Corollary 1.14. If $\Psi_0, \Psi_1 \in C^B(A)$ are both countably-generic, then $K^{\Psi_0} = K^{\Psi_1}$. Thus, if $A_0, A_1$ are $\mathcal{L}^+=\exp$-expansions of $A$ obtained as the generic models of $K^{\Psi_0}, K^{\Psi_1}$, respectively, then $A_1 \in \{A_0\}^{\text{Aut}(A)}$.

In particular, the $\aleph_0$-categorical generic expansion $A^+$ of $A$ constrained by $B$, obtained using a countably-generic ultrafilter $\Psi \in C^B(A)$, is well-defined up to isomorphism.

2. Yet Another Characterization of the Ramsey Property

In this section, we develop one more characterization of the Ramsey Property that appears to be the most “useful” one so far in that it provides a more or less straightforward architecture for induction arguments for the Ramsey Property for a given coherent class $K$ (with SAP). Essentially, we use our technology on recovering generic colorings to break down the Ramsey Property into two conditions as follows:

- $K$ has the $a$-Ramsey Property for each $\leq\text{-minimal member } a \in K$.
- $K$ is 1-simplicial – a condition that allows one to reduce a coloring $\xi : \text{Emb}(a, A) \to k$ to a coloring $\xi_0 : \text{Emb}(a_0, A) \to k$ of a proper induced substructure $a_0 \in K$ of $a$.

In the first subsection, we formulate this notion of 1-simpliciality and prove that, accompanied by the $a$-Ramsey Property for minimal $a \in K$, 1-simpliciality characterizes the Ramsey Property. Then, in two subsequent subsections, we use our new characterization to (re)prove the Ramsey Theorems of the classes of (i) all finite vertex-ordered graphs and (ii) all finite trees (with a certain kind of ordering).

We remark that our demonstrations of these Ramsey Theorems are remarkably uncomplicated induction arguments, and both completely avoid any appeals to the Hales-Jewett Theorem, which has usually been the key tool in proving Ramsey Theorems. To the best of our knowledge the earliest demonstrations of the Ramsey Property for vertex-ordered graphs and for trees are [5] and [4], respectively.

2.1. 1-Simplicial Classes. Throughout this subsection, $K$ is a coherent class with SAP, and we take $A$ to be its countable generic model. For $a \in K$, let $K(a)$ denote the set of nonempty proper substructures of $a$.

**Definition 2.1.** For the sake of poetry and the analogy with the class Ramsey Theorem (the Ramsey Property of the Fraïssé class of finite linear orders), we could say that $K$ has the Pigeonhole Property just in case it has the $a$-Ramsey Property for each minimal $a \in K$.

**Definition 2.2.** Let $a \in K$ and $c \in K(A)$, and for some $0 < k < \omega$, let $\xi : \text{Emb}(a, A) \to k$ be a generic coloring. An $A/c, \xi$-simplex is a map

$$\alpha : \text{Emb}(a, c) \to \text{Emb}(a, A) : f \mapsto \alpha_f$$

such that for each $f \in \text{Emb}(a, c)$:

- For some $a_f \in K(a)$, $\alpha_f|a_f = f|a_f$ and $\alpha_f a_f = \alpha_f a \cap c$.
  (It follows that $\|\alpha_f a\| \setminus \|c\| \neq \emptyset$.)
- $\xi(f) = \xi(\alpha_f)$.
We write \( \text{size}(\alpha) \) for the number
\[
| \{ (\alpha_f a \cap c)/\cong : f \in \text{Emb}(a, c) \} |
\]
and we write \( \Delta_{\xi}(A/c) \) for the set of all \( A/c, \xi \)-simplices.

Next, we define an \( A, \xi \)-simplex to be a mapping of the form,
\[
\alpha^\circ : c \mapsto \alpha' \in \Delta_{\xi}(A/c)
\]
for \( c \in K(A) \). Let \( \Delta_{\xi}(A) \) be the set of all \( A, \xi \)-simplices. We will say that \( K \) is 1-simplicial over \( a \) if for every generic coloring \( \xi : \text{Emb}(a, A) \to 2 \), there is an \( \alpha^\circ \in \Delta_{\xi}(A) \) such that for every \( c \in K(A) \), there are \( g \in \text{Aut}(A) \) and \( \varnothing \in K_{\varnothing}(A) \) for which \( \text{size}(\alpha^\circ|\text{Emb}(a, gc)) = 1 \).

Remark 2.3. The use of generic colorings in the formulation of 1-simpliciality seems to be unavoidable. Certainly, for any generic coloring \( \xi : \text{Emb}(a, A) \to k \), \( \Delta_{\xi}(A) \) is non-empty, but it is not at all clear that the analogous set would be non-empty for an arbitrary coloring.

**Lemma 2.4.** Suppose \( K \) has the Ramsey Property. Then \( K \) is 1-simplicial.

**Proof.** Let \( \xi : \text{Emb}(a, A) \to k \) be a generic coloring for some \( a \in K \) and \( 0 < k < \omega \). As noted above, we may choose an arbitrary \( \alpha^\circ \in \Delta_{\xi}(A) \).

Let \( c \in K(A) \) be given. Applying the (finitary) Ramsey Property, there is a model \( \varnothing_0 \in K \) such that \( \varnothing_0 \to (c)^{\varnothing_0}_a \) where \( t_a = |K(a)/\cong| \). Now, take \( v \in \text{Emb}(\varnothing_0, A) \) and \( \varnothing = v\varnothing_0 \) and consider the coloring \( \eta : \text{Emb}(a, v\varnothing) \to K(a)/\cong \) given by,
\[
\eta(f) = (\alpha^\circ f a \cap \varnothing)/\cong
\]
for each \( f \in \text{Emb}(a, \varnothing) \). Since \( \varnothing \to (c)^{\varnothing}_a \), there is an embedding \( u \in \text{Emb}(c, \varnothing) \) such that \( \eta \) is constant on \( \text{Emb}(a, uc) \). Since \( K \) is coherent, \( u \) extends to an automorphism \( g \in \text{Aut}(A) \), and clearly, \( \text{size}(\alpha^\circ|\text{Emb}(a, gc)) = 1 \), as required. \( \square \)

**Lemma 2.5.** Let \( a \in K \) such that \( K(a) \neq \emptyset \). Suppose \( K \) is 1-simplicial over \( a \), and suppose that \( K \) has the \( a_0 \)-Ramsey Property for each \( a_0 \in K(a) \). Then \( K \) has the \( a \)-Ramsey Property.

**Proof.** Let \( \xi : \text{Emb}(a, A) \to 2 \) be a generic coloring. We must show that there is an elementary embedding \( e : A \to A \) such that \( \xi \) is constant on \( \text{Emb}(a, eA) \). Using the genericity of \( \xi \), it will suffice to construct an \( \ell \)-embedding \( w : A \hookrightarrow A \) such that \( \xi \) is constant on \( \text{Emb}(a, wX[A^\alpha_X]) \) for every \( X \subset_{\text{fin}} A \).

As \( K \) is 1-simplicial over \( a \), let \( \alpha^\circ \in \Delta_{\xi}(A) \) such that for every \( c \in K(A) \) there is a \( g \in \text{Aut}(A) \) for which \( \text{size}(\alpha^\circ g e) = 1 \). By the standard Pigeonhole Principle, there is an \( \ell \)-embedding \( h : A \hookrightarrow A \) such that for every \( Y \subset_{\text{fin}} A \), for all \( f \in \text{Emb}(a, hY[A^\alpha_Y]) \),
\[
a_f = a_f hY[A^\alpha_Y] a \cap hY[A^\alpha_Y] \cong a_0.
\]
For \( Y \subset_{\text{fin}} A \), we define a coloring \( \xi_Y : \text{Emb}(a_0, hY[A^\alpha_Y]) \to 3 \) by
\[
\xi_Y(f_0) = \begin{cases} 
\xi(f) & f_0 = a_f hY[A^\alpha_Y] a_0, f \in \text{Emb}(a, hY[A^\alpha_Y]) \\
2 & \text{otherwise}.
\end{cases}
\]
Now, let \( b \in K(A) \) be given, and suppose \( \text{Emb}(a, b) \neq \emptyset \). First, we choose \( |b| \subset Y \subset_{\text{fin}} A \) such that for any coloring \( \eta : \text{Emb}(a_0, A_Y) \to 3 \), there is an embedding \( u \in \text{Emb}(b, A_Y) \) such that \( \eta \) is constant on \( \text{Emb}(a_0, ub) \). Next, we choose some embedding \( u_b \in \text{Emb}(b, hY[A^\alpha_Y]) \).
such that $\zeta_Y$ is constant on $\text{Emb}(a_0, u_b b)$. Now, since $\text{Emb}(a, b)$ is nonempty, there must be $f \in \text{Emb}(a, u_b b)$ so that $\alpha_{f_{[AV]}}^b |a_0 = f |a_0$, implying that

$$\zeta_Y(f |a_0) = \xi(f) \in \{0, 1\};$$

thus, $\zeta_Y|\text{Emb}(a, ub) \neq 2$. It follows that for all $f_0, f_1 \in \text{Emb}(a, u_b b)$,

$$\xi(f_0) = \xi(\alpha_{f_0}^{b_{[AV]}})$$

$$= \zeta_Y(f_0 |a_0)$$

$$= \zeta_Y(f_1 |a_0)$$

$$= \xi(\alpha_{f_1}^{b_{[AV]}})$$

$$= \xi(f_1)$$

That is, $\xi$ is constant on $\text{Emb}(a, u_b b)$. Let us write $Y_b \subsetfin A$ for the finite subset of $A$ used above in relation to $b$. To conclude the demonstration, we define $w : A \rightsquigarrow A$ by taking $A_X = Y_{A|X}$ and $w_X = u_{A|X}$ for each $X \subsetfin A$. \qed

**Proposition 2.6.** Suppose $K$ has the Pigeonhole Property (i.e. $a$-RP for each minimal $a \in K$) and $K$ is 1-simplicial. Then $K$ has the Ramsey Property.

**Proof.** The proof is (would be) by induction on $|a|$, showing that for each $n < \omega$ and each $a \in K$, if $|a| \leq n$, then $K$ has the $a$-Ramsey Property. The base case(s) is covered by the assumption that $K$ has the $a$-Ramsey Property for every minimal $a \in K$. Then, the induction step is covered by the previous lemma. \ql

**2.2. Example: Ordered graphs.** In this subsection, let $OG$ denote the Fraïssé class of vertex-ordered finite graphs. Let $G = (G, R^G, <^G)$ be the random ordered-graph, and let $1_{OG}$ be the graph with vertex set $\{\star\}$ (a singleton), necessarily no edges, and necessarily trivial order relation. We note that, up to isomorphism, $1_{OG}$ is the unique minimal member of $OG$.

**Theorem 2.7.** The class $OG$ of finite vertex-ordered graphs has the Ramsey Property.

**Proof.** We just need to establish the two claims A and B below:

**Claim (A).** $OG$ has the $1_{OG}$-Ramsey Property.

**Proof.** Let $\xi : \text{Emb}(1_{OG}, G) \rightarrow 2$ be a generic coloring; that is, $\xi$ is a really map $V^G \rightarrow 2$ such that for all $a \leq b \in OG(G)$ and $g \in \text{Aut}(G)$, if $\|b\| = \|a\| \cup \{\mathbf{b}\}$ and $\xi(g(a)) = \xi(a)$ for all $a \in \|a\|$, then there is a $g' \in \text{Aut}(G)$ such that $g' [a = g |a$ and $\xi(g'(b)) = \xi(b)$.

Now, let $(c_n)_{n<\omega}$ be a chain of finite substructures of $G$ such that $\bigcup_n c_n = G$, $|c_0| = 1$, and $|c_{n+1}| = |c_n| + 1$ for all $n < \omega$. Further, for each $n < \omega$, let $\Gamma^n_0$ be the set of embeddings $u \in \text{Emb}(c_n, G)$ such that $\xi(u(a)) = 0$ for all $a \in \|c_n\|$, and define $\Gamma_n^1$ similarly. Clearly, if $m \leq n$, $t < 2$, $u \in \Gamma^n_t$, then $u |c_m \in \Gamma^t_m$. $\Gamma_n^t$ is non-empty for every $n < \omega$. $\Gamma_n^t$ is non-empty for every $n < \omega$.

**Proof of sub-claim.** Without loss of generality, we assume that $t = 0$ and $\Gamma_0^0$ is non-empty; let $u \in \Gamma_0^0$. Take $\{c\} = \|c_{m+1}\| \setminus \|c_m\|$, and let $p(x, uc_m) = \text{qftp}(c/uc_m)$. Consider any $b \in OG$ and any embedding $v : b \rightarrow G$ such that $\|vb\| \subseteq p(G, uc_m)$; then $\xi(b) = 1$ for every $b \in \|vb\|$. That such embeddings exist for every $b \in OG$ is clear upon some reflection. For any $n < \omega$, then, let $v_n : c_n \rightarrow G$ such that $\|v_n c_n\| \subseteq p(G, c_m)$; then $v_n \in \Gamma_{n+1}^1$. \qed
Now, let \( \Gamma_n = (\Gamma_n^0 \cup \Gamma_n^1) / \sim_n \) where \( u \sim_n v \) just in case there is an automorphism \( g \in Aut(\mathcal{G}^2) \) such that \( v = g \circ u \). Ordering \( \Gamma = \bigcup_n \Gamma_n \) in the obvious way (by factoring \( \subseteq \) through the \( \sim_n \)'s), we are left with an infinite, finitely-branching tree \( \Gamma \). By König’s Lemma, there is an infinite branch \((v_n/\sim_n)_{n<\omega}\), and it is not difficult to see that this branch can be converted into an (elementary) embedding \( w : \mathcal{G} \to \mathcal{G} \) such that \( \xi \) is constant \( w[\mathcal{G}] \). This completes the proof of the lemma.

\[ \square \]

**Claim (B).** \( OG \) is 1-simplicial.

**Proof.** Let \( a \in OG \), and let \( \xi : \text{Emb}(a, \mathcal{G}) \to 2 \) be a generic coloring. By Proposition ###, we may assume that for all \( A \subset_{\text{fin}} \mathcal{G} \), \( acl^{\mathcal{G}}(A) = acl^{\mathcal{G}}(A) = A \). Fix \( z = \max_{<\omega} \|a\| \), and let \( a_0 = a([a] \setminus \{z\}) \). To define our \( A, \xi \)-simplex, we consider one graph \( c \in OG(\mathcal{G}) \) in isolation, defining \( A/c, \xi \)-simplex; for this, we consider an embedding \( f \in \text{Emb}(a, c) \). Let \( c_0 = c([c] \setminus \{f(z)\}) \). As \( Th(\mathcal{G}^2) \) is \( \aleph_0 \)-categorical and preserves the algebraic-closure operator of the reduct \( Th(\mathcal{G}) \), then \( p(x) = qftp^{\mathcal{G}}(f(z)/c_0) \) has infinitely many realizations, so we may choose \( z_f \in p(\mathcal{G}) \ \parallel \ c \) and set \( a^c_f = f[a_0] \cup \{(z, z_f)\} \). Defining \( a^c_f \) similarly for each \( f \in \text{Emb}(a, c) \), we recover an \( A/c, \xi \)-simplex \( \alpha^c \) with \( \text{size}(\alpha^c) = 1 \). Defining \( \alpha^c \) similarly for all \( c \in OG(\mathcal{G}) \) completes the proof.

\[ \square \]

2.3. **Example: Trees.** The language of trees \( \mathcal{L} \) has the following signature: \( \sqcap, \prec, E \) are binary relation symbols. For a finite tree \( W \subset_{\text{fin}} \omega^{<\omega} \) (\( W \) is a finite set of sequences, closed under taking initial segments, we define a structure \( a_W \) as follows:

- \( \|a_W\| = W \).
- \( \sqcap^{a_W} \) is the (proper) initial segment relation.
- \( E^{a_W} = \{(w, w') \in W : |w| = |w'|\} \).
- \( <^{a_W} \) is defined as follows: If \( |w| < |w'| \), then \( w <^{a_W} w' \); if \( |w| = |w'| \), then \( w <^{a_W} w' \) if and only if \( w <_{\text{lex}} w' \) (as \( n \)-tuples of integers where \( n = |w| = |w'| \)).

We define \( \text{Tree} \) to be the closure of \( \{a_W : W \subset_{\text{fin}} \omega^{<\omega} \text{ a non-empty tree}\} \) under isomorphisms of \( \mathcal{L} \)-structures, and members of \( \text{Tree} \) will be called finite leveled trees.

Define \( 1_{\text{Tree}} = a_{\{\emptyset\}} \), where \( \{\emptyset\} \) is the tree consisting of just the empty function \( \emptyset \to \omega \) (the tuple of length 0). As usual, let \( A \) be the countable generic model of \( \text{Tree} \). (We must remark, here, that \( A \) is not, in fact, a tree – it has no root, and its branches have order-type \( \mathbb{Q} \).) We also note that \( \text{Tree} \) is a coherent class, but it is not a Fraïssé class – even though \( A \) is the Fraïssé limit of a slightly broader class of finite “forests” expressed as \( \mathcal{L} \)-structures.

**Theorem 2.8.** The class \( \text{Tree} \) of finite leveled trees has the Ramsey Property.

**Proof.** Again, we need only verify the two claims A and B below.

**Claim (A).** \( \text{Tree} \) has the \( 1_{\text{Tree}} \)-Ramsey Property.

**Proof.** We show that \( \text{Tree} \) has \( 1_{\text{Tree}} \)-ARP. Let \( \xi_0 : \text{Emb}(1_{\text{Tree}}, A) \to 2 \) be an arbitrary coloring, and let \( \xi : \text{Emb}(1_{\text{Tree}}, A) \to 2 \) be the generic coloring constrained by \( \xi_0 \). Now, to demonstrate \( 1_{\text{Tree}} \)-ARP, it is enough to construction an embedding \( \eta : \omega^{<\omega} \to A \) such that for all \( w, w' \in \omega^{<\omega} \):

- \( \xi(\eta(w)) = \xi(\eta(w')) \);
- if \( |w| = |w'| \), then \( (\eta(w), \eta(w')) \in E^A \).
• if $|w| = |w'|$ and $w \prec_{\text{lex}} w'$, then $\eta(w) \prec^A \eta(w')$.

As in the case of vertex-ordered graphs, we note that an embedding $f \in \Emb(1_{\text{Tree}}, A)$ is nothing more than an element of $W_f$, and we modify the definition of $|w|$ slightly: if $b \in \Tree$ and $g \in \Aut(A)$, if $|b| = |a| \cup \{b\}$ and $\xi(g(a)) = \xi(a)$ for each $a \in |a|$, then there is a $g' \in \Aut(A)$ such that $g'|a = g|a$ and $\xi(g'(b)) = \xi(b)$.

Now, let us fix a bijection $\beta : \omega \to \omega^\omega$ such that $\beta(0) = 0$, and for every $0 < n < \omega$, $W_n := \{\beta(i) : i < n\}$ is a sub-tree and $\beta(n) = \beta(i)^t$ for some $i < n$ and $t < \omega$. For each $n < \omega$, let $\Gamma_n$ be the set of embeddings $f \in \Emb(a_{W_n}, A)$ such that $\xi(f(w)) = 0$ for all $w \in W_n$, and define $\Gamma \upharpoonright _i$ similarly. Once again, if $m \leq n$, $t < 2$, $u \in \Gamma_m$, then $u|\alpha_{W_m} \in \Gamma_t$.

Sub-claim. Given $m < \omega$, if $\Gamma_{m+1}$ is empty, then $\Gamma_{n-t}$ is non-empty for every $n < \omega$.

Proof of claim. Without loss of generality, we assume that $t = 0$ and $\Gamma_0$ is non-empty; let $f, g, h \in \Gamma_0$. Let $p(x, a_{W_m}) = qftp^{aw_m+1}(\beta(n)/a_{W_m})$. Now, for any $b \in \Tree$ and any embedding $v : b \to A$ such that $|vb| \subseteq p(A, u_{a_m})$, we have $\xi(b) = 1$ for every $b \in |vb|$. Thus, $\Gamma_1$ is infinite for all $n < \omega$.

Now, let $\Gamma_n = (\Gamma_n \cup \Gamma_n)/\sim_n$ where $f \sim_n \gamma$ just in case there is an automorphism $g \in \Aut(A^\xi)$ such that $v = g \circ u$. Ordering $\Gamma = \bigcup_n \Gamma_n$ in the obvious way (by factoring through the $\sim$’s), we are left with an infinite, finitely-branching tree $\Gamma$. By König’s Lemma, there is an infinite branch $(f_n/\sim_n)_{n<\omega}$, and it is not difficult to see that this branch can be converted into an (elementary) embedding $\eta : \omega^\omega \to A$ such that $\xi$ is constant $\eta[\omega^\omega]$. This is sufficient for $1_{\text{Tree}}$-ARP against $\xi_0$, so this completes the proof of the lemma.

Claim (B). $\text{Tree}$ is 1-simplicial.

Proof. Let $\xi : \Emb(a_W, A) \to 2$ be a generic coloring for some finite tree $W \subset_{\text{fin}} \omega^\omega$ such that $|W| > 1$. Let $w_0$ be a maximal element of $W$ (a leaf), and let $W = W \setminus \{w_0\}$. For the moment, we assume that $a_W \not\equiv E(w_0, w)$ for every $w \in W$, but we will rectify this silliness (meant for clarity) shortly.

Let $MC(A)$ denote the set of all maximal $\Box^A$-chains in $A$; let $\mathcal{B}^* ϴ$ be an $\aleph_1$-saturated elementary extension of $A^\xi$, and let $v^* : MC(A) \to B$ be such that for each $C \in MC(A)$, $v^*(C)$ is a realization of the partial type $\pi_C(x) = \{a \in C : a \in C\}$. We define a coloring $\zeta : \Emb(a_W, A) \to 2$ as follows: Let $f \in \Emb(a_W, A)$; arbitrarily, choose a chain $C \in MC(A)$ such that

$$B(\{f(a_W) \cup \{v^*(C)\}) \cong a_W$$

then set $\zeta(f) = \xi(f \cup \{(w_0, v^*(C))\})$. Since $\text{Tree}$ the $a_W$-Ramsey Property, there is an $\ell$-embedding $h : A \sim A$ such that $\zeta$ is constant on $\Emb(a_W, h_A[A_X])$ for every $X \subset_{\text{fin}} A$, and this suffices for 1-simpliciality over $a_W$.

The rectification: let $n = \max\{|w| : w \in W\}$; then $W_n = \{w \in W : |w| = n\}$ is an $E^w$-class consisting of leaves. Instead of $W = W \setminus \{w_0\}$ as above, we take $W = W \setminus W_n$, and we modify the definition of $\zeta$ as follows: Let $f \in \Emb(a_W, A)$; arbitrarily, choose chains $C_w \in MC(A)$ for each $w \in W_n$ such that

$$B(\{f(a_W) \cup \{v^*(C_w)\}_{w \in W_n}) \cong a_W$$

then set $\zeta(f) = \xi(f \cup \{(w, v^*(C_w)) : w \in W_n\})$. The remainder of the argument goes through unchanged. □
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